SCALAR CURVATURE, METRIC DEGENERATIONS AND THE STATIC VACUUM EINSTEIN EQUATIONS ON 3-MANIFOLDS, I.

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Contents

0. Introduction

In this paper, we prove that degenerations of sequences of Yamabe metrics on 3-manifolds are modeled or described by solutions to the static vacuum Einstein equations. One underlying motivation to understand such degenerations is the question of existence of constant curvature metrics on 3-manifolds, in other words with the geometrization conjecture of Thurston [Th2]. An approach towards resolving this conjecture via study of Yamabe metrics is outlined in [An1].

Let $\mathbb{M}$ denote the space of all smooth Riemannian metrics on a closed, oriented 3-manifold M, and $\mathbb{M}_1$ the subset of metrics satisfying $\mathrm{vol}_g M = 1$. Define the total scalar curvature or Einstein-Hilbert action $\mathcal{S} : \mathbb{M} \to \mathbb{R}$ by

$$\mathcal{S}(g) = v^{-1/3} \int_M s_g dV_g, \tag{0.1}$$

where s_g is the scalar curvature of the metric g, dV_g is the volume form associated with g and v is the volume of (M, g). The critical points of $\mathcal{S}$ are *Einstein metrics*, i.e. metrics satisfying the equation

$$-\nabla \mathcal{S}|_g = v^{-1/3} z = v^{-1/3}\left(r - \frac{s}{3}\cdot g\right) = 0, \tag{0.2}$$

where r is the Ricci curvature of g and z is the traceless Ricci curvature. In dimension 3, (and only in this dimension), Einstein metrics are exactly the metrics of constant curvature.

There is a well-known minimax procedure to obtain critical values of $\mathcal{S}$. First, the solution to the Yamabe problem [Y], [Tr], [Au1], [Sc1] implies that in the conformal class $[g]$ of any metric $g \in \mathbb{M}_1$, there is a metric $\bar{g} \in \mathbb{M}_1$ which realizes the infimum $\mu[g]$ of $\mathcal{S}|_{[g]\cap\mathbb{M}_1}$, i.e.

$$s_{\bar{g}} = \mu[g] \equiv \inf\ \mathcal{S}|_{[g]\cap\mathbb{M}_1}. \tag{0.3}$$

Such metrics are called *Yamabe metrics*. Let $\mathcal{C}$ denote the space of Yamabe metrics in $\mathbb{M}$, (of arbitrary volume), and $\mathcal{C}_1$ the subset of unit volume Yamabe metrics; any metric in $\mathcal{C}$ can be scaled uniquely to a metric in $\mathcal{C}_1$. In dimension 2, the space $\mathcal{C}_1$, when divided by the action of the diffeomorphism group, corresponds to the Teichmüller or moduli space of a surface, and the functional $s_{\bar{g}} : \mathcal{C}_1 \to \mathbb{R}$ is a constant function, (by the Gauss-Bonnet theorem). In dimension 3, (and above), this is an infinite dimensional space, and the functional is not constant. A simple but important comparison argument of Aubin [Au1] implies that, for any conformal class [g] on any M,

$$\mu[g] \leq \mu(S^3, g_{can}), \tag{0.4}$$

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where g_{can} is the canonical constant curvature metric on S^3 of volume 1. Define the *Sigma constant* $\sigma(M)$ of M by

$$\sigma(M) = \sup_{\mathcal{C}_1} \mu[g]. \tag{0.5}$$

This is a smooth invariant of the 3-manifold M, which should act very much like an Euler characteristic for 3-manifolds, when appropriately normalized. In case $\sigma(M) \leq 0$, it is easy to prove, c.f. [Bes, Prop.4.47], that any Yamabe metric $g_o \in \mathcal{C}_1$ such that $s_{g_o} = \sigma(M)$ is Einstein. In case $\sigma(M) > 0$, this has been conjectured to be true, c.f. [Bes, Remark4.48], but remains still unknown.

Of course, an arbitrary closed, oriented 3-manifold does not admit an Einstein metric; this is the case if for instance M has an essential 2-sphere. Thus, if M is an arbitrary closed 3-manifold and if $\{g_i\}$ is a maximizing sequence of unit volume Yamabe metrics on M, that is

$$s_{g_i} \rightarrow \sigma(M), \tag{0.6}$$

then in general this sequence must degenerate in some manner.

More specifically, if $\{g_i\}$ satisfies the uniform curvature bound

$$\int_M |z_{g_i}|^2 dV_{g_i} \leq \Lambda, \tag{0.7}$$

for some $\Lambda < \infty$, then as outlined in [An1], the structure of a suitable subsequence of $\{g_i\}$ and its limit can be used to geometrize the 3-manifold M, (at least when $\sigma(M) \leq 0$). This is accomplished essentially via the Cheeger-Gromov theory of convergence and collapse of Riemannian manifolds [C], [G, Ch.8], [CG1,2]. Since an arbitrary 3-manifold cannot be given a geometric structure, a maximizing sequence $\{g_i\}$ of unit volume Yamabe metrics must in general satisfy

$$\int_M |z_{g_i}|^2 dV_{g_i} \rightarrow \infty, \qquad \text{as } i \rightarrow \infty. \tag{0.8}$$

For the purposes of this paper, we will say that a sequence $\{g_i\}$ satisfying (0.8) *degenerates.* One then would like to relate the geometry of a degenerating sequence of Yamabe metrics with some topological structure on the underlying manifold M that is the cause for the degeneration, c.f. [An1, §4-6] for further discussion.

The main purpose of this paper is to show that the degenerations of such a sequence $\{g_i\}$ are described by solutions to the static vacuum Einstein equations. Further, under natural conditions, the degenerations correspond to non-trivial solutions of these equations.

Some further background is needed to explain this; we refer to §1-§2 for further details. Let L be the linearization of the scalar curvature function at g given by

$$L(\alpha) = \frac{d}{dt} s(g + t\alpha)\Big|_{t=0} = -\Delta \operatorname{tr} \alpha + \delta\delta\alpha - \langle r, \alpha \rangle, \tag{0.9}$$

c.f. [Bes, Ch.1K]. The L^2 adjoint L^* of L is given by

$$L^*(f) = D^2 f - \Delta f \cdot g - f \cdot r. \tag{0.10}$$

For any $g \in \mathcal{C}$, one thus has a natural splitting

$$T_g\mathbb{M} = T_g\mathcal{C} \oplus N_g\mathcal{C}, \tag{0.11}$$

where

$$T_g\mathcal{C} = \{\alpha \in T_g\mathbb{M} : L(\alpha) = const\}, \qquad N_g\mathcal{C} = \left\{\beta \in T_g\mathbb{M} : \beta = L^*(q), \int q = 0\right\},$$

are the (formal) tangent and normal spaces to $\mathcal{C}$ in $\mathbb{M}$.

In particular, the L^2 gradient $\nabla\mathcal{S} = -v^{-1/3}z$ of the scalar curvature functional $\mathcal{S}$ splits at g as

$$z = z^T + z^N, \tag{0.12}$$

where $-v^{-1/3}z^T$ is the gradient of the scale invariant functional $v^{2/3}\cdot s = \mathcal{S}|_{\mathcal{C}}$ on $\mathcal{C}$.

Another natural splitting of $T\mathbb{M}$, differing from (0.10) by just a 1-dimensional factor, is

$$T_g\mathbb{M} = \mathrm{Ker}\ L \oplus \mathrm{Im}\ L^*. \tag{0.13}$$

As in (0.12), we have then an L^2 orthogonal sum

$$z = \xi + L^*f, \tag{0.14}$$

for some $\xi \in \mathrm{Ker}\ L$ and $f \in C^\infty(M)$. We set

$$u = 1 + f. \tag{0.15}$$

Next, we describe briefly the *static vacuum Einstein equations.* These are equations

$$\begin{aligned} hr &= D^2h, \\ \Delta h &= 0, \end{aligned} \tag{0.16}$$

for a pair (g, h) consisting of metric g and positive harmonic potential function h, defined on an open 3-manifold N. These equations have been extensively studied in classical general relativity. They imply that the 4-manifold $X = N \times_h S^1$, with warped product metric $g_X = g_N + h^2d\theta^2$ is Ricci-flat, i.e. a vacuum solution to the Einstein equations in dimension 4, c.f. §1.3.

Of course, a flat metric, with h an affine function, is a solution of (0.16). It is proved in the Appendix, c.f. also Theorems 3.2 and 3.3, that the only complete solution to the static vacuum equations (0.16) with potential $h > 0$ everywhere is a flat metric, with $h = const$. The most important or "canonical" (non-trivial) solution to the static vacuum equations is the Schwarzschild metric g_s,

$$g_s = (1 - 2mt^{-1})^{-1}dt^2 + t^2ds^2_{S^2}, \tag{0.17}$$

defined on $[2m, \infty) \times S^2$, with $h = (1 - 2mt^{-1})^{1/2}$; here the mass m is a positive constant. This metric is asymptotically flat, (for large t), and the locus $\Sigma = \{h = 0\} = \{t = 1\}$ is a round totally geodesic 2-sphere, of radius $2m$. Physically, this represents the surface of an (idealized) static black hole.

Finally, given a sequence $\{g_i\} \in \mathbb{M}_1$, a sequence $r_i \in \mathbb{R}$ converging to 0, and points $x_i \in M$, the blow-up sequence is the pointed sequence of Riemannian manifolds $(M, \bar{g}_i, x_i)$ with $\bar{g}_i = r_i^{-2}\cdot g_i$. For instance, in case the curvature tensor R_i of (M, g_i) is everywhere bounded by r_i^{-2}, the Cheeger-Gromov theory implies that a subsequence of the pointed sequence $(M, \bar{g}_i, x_i)$ either converges, modulo diffeomorphisms, uniformly on compact subsets, to a limit complete Riemannian manifold $(N, \bar{g}, x)$, or collapses along a sequence of F-structures to a lower dimensional space.

A simplified version of the main result of this paper is the following:

Theorem A. *Let $\{g_i\}$ be a sequence of unit volume Yamabe metrics on a closed oriented 3-manifold M, with $s_{g_i} \geq -s_o \geq -\infty$, satisfying the following condition:*
(i). (Non-Collapse). There is a constant $\nu_o \geq 0$ such that

$$\mathrm{vol}\, B_x(r) \geq \nu_o r^3, \tag{0.18}$$

for any geodesic ball $B_x(r) \subset (M, g_i), r \leq \mathrm{diam}(M, g_i)$.

(I). *Then given any sequence of points $x_i \in (M, g_i)$ for which $\int_{B_{x_i}(r_o)} |z_{g_i}|^2 dV_{g_i} \to \infty$, for some $r_o > 0$, there is a blow-up sequence (M, g_i', x_i),*

$$g_i' = \rho_i^{-2} \cdot g_i, \rho_i \to 0,$$

such that a subsequence converges to a locally defined solution of the static vacuum Einstein equations.

(II). *In addition to (i), suppose the following conditions hold:*
(ii). There is a constant $K < \infty$, such that

$$\int_M |z_{g_i}^T|^2 dV_{g_i} \leq K. \tag{0.19}$$

(iii). For u as in (0.15), there is a constant $\delta_o > 0$ such that the sequence $\{g_i\}$ degenerates on the domain $U_{\delta_o} = \left\{ x \in (M, g_i) : \dfrac{|u_i(x)|}{\sup |u_i|} \geq \delta_o \right\}$, i.e.

$$\int_{U_{\delta_o}} |z_{g_i}|^2 dV_{g_i} \to \infty. \tag{0.20}$$

Then there are points $y_i \in U_{\delta_o}$ with $|z_{g_i}|(y_i) \to \infty$, and a blow-up sequence (M, g_i', y_i) defined as above, such that a subsequence converges to a non-trivial, (in particular non-flat), locally defined solution to the static vacuum Einstein equations.

We first make some comments on the hypotheses and conclusions. Condition (i) is used to rule out the possibility of collapse of the blow-up sequence. This condition will be weakened in §3, but some version of it is essential for Theorem A. Without any lower bound on the volumes of (small) geodesic balls, one cannot expect blow-ups to converge at all, see §4.1 for further discussion. Theorem A(I) gives then a general relationship between degenerations of Yamabe metrics and solutions, possibly trivial, of the static vacuum equations.

Conditions (ii) and (iii) are required to obtain a *non-trivial* limit solution. For instance, in case $\sigma(M) < 0$, condition (ii) prevents the function u, which basically serves as the potential function h in the static vacuum solution (0.16), from going to 0 in L^2. The condition (ii) is not as strong an assumption as it might at first appear. For example, it is proved in Theorem 2.10 that if $\{g_i\}$ is an arbitrary sequence of unit volume Yamabe metrics on M for which there exist points $x_i \in M$ and arbitrary numbers $r_o \geq 0$, $R_o < \infty$, such that

$$\int_{B_{x_i}(r_o)} |r_{g_i}|^2 \leq R_o, \qquad \mathrm{vol}\, B_{x_i}(r_o) \geq R_o^{-1}, \tag{0.21}$$

then (0.19) holds, with K depending only on r_o, R_o.

Note that (0.21) requires only that $\{g_i\}$ have locally bounded ($L^{2,2}$) geometry in some ball of small but fixed size in M, and yet it implies the global conclusion (0.19). This novel feature of some of the global aspects of Yamabe metrics plays a central role in this paper.

Finally, the condition (iii) that $\{g_i\}$ degenerates on U_{δ_o}, for some $\delta_o > 0$, and not just somewhere on (M, g_i) as in (0.8), is also essential to obtain a non-trivial blow-up limit solution. Examples discussed in §6 will illustrate the necessity of this condition.

A much more precise formulation of Theorem A will be proved in Theorem 3.10 below, where the base points y_i and scale factors ρ_i are constructed from the geometry of (M, g_i). Further, it is shown in §5 that the limit solutions are complete in a natural sense and some initial results on their asymptotic geometry are obtained.

Theorem A implies that blow-up limits of degenerations in U_{δ_o} of sequences of Yamabe metrics satisfy a strong, in particular a determined, set of equations. This is quite surprising, since Yamabe metrics themselves satisfy no such rigid equations; the equation defining a Yamabe metric is highly underdetermined. Note also that Theorem A holds for quite general sequences of Yamabe metrics; for example no assumption is made that the sequence $\{g_i\}$ is a maximizing sequence for $\mathcal{S}|_{\mathcal{C}}$.

It has long been known that static vacuum solutions on an open 3-manifold N are closely tied with 2-spheres in N, occurring at the event horizon or boundary Σ of a black hole in general relativity. This is already seen in the Schwarzschild metric (0.17), and is apparent in the work of Hawking; c.f. [HE, Ch.9.3] and §1.3 for further discussion. More generally, many solutions of the static vacuum equations are asymptotically flat, and thus have natural 2-spheres near infinity. These remarks illustrate a basic tie relating the degeneration of Yamabe metrics with the underlying topology of the manifold mentioned above.

On the other hand, the first two examples discussed in §6 show that the restriction to U_{δ_o} in Theorem A is essential. If $\{g_i\}$ degenerates only on the complement of U_{δ_o}, for any given $\delta_o > 0$, then the blow-ups describing the degeneration may satisfy the static vacuum equations only trivially; see § 3.2. We also present examples in §6 of degenerating sequences which do satisfy the hypotheses of Theorem A(II), showing that this result is indeed applicable.

It is certain global features of Yamabe metrics which lead to the necessity of condition (iii) in Theorem A. In §7, we consider the relation of this condition with the condition that a sequence of Yamabe metrics is Palais-Smale for the functional $\mathcal{S}|_{\mathcal{C}}$. (Recall that a sequence is Palais-Smale for a functional if the gradient tends to 0, in a suitable (weak) topology; this is not to be confused with Condition C of Palais-Smale, which is much too strong to be of use here). In §4.2, it is shown that condition (iii) follows for sequences of Yamabe metrics satisfying a natural strengthened version of the definition of Palais-Smale sequence for $\mathcal{S}|_{\mathcal{C}}$, at least when $\sigma(M) < 0$.

In any case, in order to effectively describe the structure of degenerations of sequences of Yamabe metrics on all of M, (not only on U_{δ_o}), one needs to restrict to special classes of sequences. In addition to the reason above, this is also needed in case the sequence collapses, as discussed in §4.1. Some such preferred sequences are considered already in [An2, §8], and we will discuss these and related special sequences in more detail in [AnII].

This paper is partly intended as the third in a series of works on the geometrization conjecture of Thurston, the previous works being [An1, 2].

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1. Background Material.

§1.1. Throughout the paper, M will denote a compact, connected, oriented 3-manifold, without boundary. Let $\mathbb{M}$ denote the space of smooth (C^∞) Riemannian metrics on M. This is an open cone, (the positive definite cone), in the linear space $S^2(M)$ of smooth symmetric bilinear forms. For any $g \in \mathbb{M}$, one has a natural identification

$$T_g\mathbb{M} = S^2(M), \tag{1.1}$$

of the tangent space of $\mathbb{M}$ at g. Let $\mathcal{M}$ denote the space of isometry classes of all C^∞ Riemannian metrics; thus $\mathcal{M}$ is the quotient of $\mathbb{M}$ by the action of the diffeomorphism group $\mathrm{Diff}(M)$ on $\mathbb{M}$. It is well known, [Bes, Ch.4B], that the tangent space $T_g\mathbb{M}$ splits as

$$T_g\mathbb{M} = \mathrm{Im}\ \delta^* \oplus \mathrm{Ker}\ \delta, \tag{1.2}$$

where $\delta : S^2(M) \to \Omega^1(M)$ is the divergence operator (w.r.t. g) on $S^2(M)$, given by $\delta\ (\alpha) = -(D_{e_i}\alpha)(e_i, \cdot)$; here $\{e_i\}$ is an orthonormal basis of TM. The operator $\delta^* : \Omega^1(M) \to S^2(M)$ is the formal adjoint of δ, given by $\delta^*(\omega) = \frac{1}{2}\mathcal{L}_\omega g; \mathcal{L}$ denotes the Lie derivative, and we are identifying vector fields and 1-forms via the metric. In the splitting (1.2), the factor Im δ^* is the tangent space to the orbit of $\mathrm{Diff}(M)$ at g, while the factor Ker δ is naturally identified with the tangent space of the quotient $\mathcal{M}$, at least when g has no isometries.

There is a natural (weak) Riemannian metric on $\mathbb{M}$, the L^2 metric, given for $\alpha, \beta \in T_g\mathbb{M}$ by

$$\langle \alpha, \beta \rangle \ = \int_M \mathrm{tr}_g(\alpha \circ \beta) dV_g, \tag{1.3}$$

where α, β are considered as linear maps of TM, via g, and dV_g denotes the volume form of g. It is easily verified that this L^2 metric is invariant under the action of $\mathrm{Diff}(M)$, and thus passes to the quotient to give the L^2 metric on $\mathcal{M}$. Further, the splitting (1.2) is orthogonal w.r.t. the L^2 metric.

The subsets of $\mathbb{M}$ and $\mathcal{M}$ consisting of metrics of volume 1 on M will be denoted by $\mathbb{M}_1$ and $\mathcal{M}_1$. The discussion above is also valid for $\mathbb{M}_1$ and $\mathcal{M}_1$.

The (weak) L^2 Riemannian metric does not give a good (smooth) topology to $\mathbb{M}$. For our purposes, it is natural to put the $L^{2,2}$ topology on $\mathbb{M}$, i.e. the topology given by the $L^{2,2}$ Riemannian

metric, defined on each $T_g\mathbb{M}$ by

$$||h||_{L^{2,2}} = \int_M |h|^2 + |Dh|^2 + |D^2h|^2 \, dV_g^{\ 1/2}, \tag{1.4}$$

for $h \in T_g\mathbb{M}$. Here the norms $|\cdot|$, and covariant derivative D are taken with respect to the metric $g \in \mathbb{M}$. Thus, the $L^{2,2}$ distance between two metrics g, g' is the infimum of the lengths of smooth curves joining g and g'; the length of a curve γ is defined by

$$L(\gamma) = \int \left|\left|\frac{d\gamma}{dt}\right|\right|_{L^{2,2}(\gamma(t))} dt. \tag{1.5}$$

This norm corresponds formally to the Sobolev space $L^{2,2}$ of functions with two weak derivatives in L^2. In fact, it is not difficult to verify, c.f. [E, p.21], that the $L^{2,2}$ metric topology above induces the same topology as the $L^{2,2}$ topology defined by local coordinates, i.e.

$$\mathrm{dist}_{L^{2,2}}(g, g') < \varepsilon \ \Leftrightarrow \ |g_{ij} - g'_{ij}|_{L^{2,2}} < \varepsilon', \tag{1.6}$$

where $|\cdot|_{L^{2,2}}$ is the Sobolev topology on functions on bounded domains in $\mathbb{R}^3$, and the components g_{ij}, g'_{ij} are taken with respect to an arbitrary fixed coordinate atlas of M; here $\varepsilon' = \varepsilon'(\varepsilon)$ is small if ε is small, and vice versa. Although the two topologies defined by (1.5) and (1.6) are the same, the two metrics are very different in the large on $\mathbb{M}$.

The Sobolev embedding theorem for $L^{2,2}$ functions in dimension 3 reads

$$L^{2,2} \subset L^{1,6} \subset C^{1/2}; \tag{1.7}$$

this is understood to apply to functions of compact support say in the unit ball in $\mathbb{R}^3$. Here C^α denotes the space of Hölder continuous functions with Hölder exponent α, $L^{p,q}$ the Sobolev space of functions with p weak derivatives in L^q. Thus, at any $g \in \mathbb{M}$, one has an estimate of the form

$$c_s(g)||\alpha||_{C^{1/2}(g)} \leq ||\alpha||_{L^{2,2}(g)}, \tag{1.8}$$

for any 2-tensor α. The Sobolev constant c_s depends strongly on the metric g, so that the estimate (1.8) is not uniform on $\mathbb{M}$.

It is useful to compare the $L^{2,2}$ metric with a somewhat weaker metric. Thus, define the $T^{2,2}$ metric to be the Riemannian metric given on each $T_g\mathbb{M}$ by the norm

$$||h||_{T^{2,2}} = \int_M |h|^2 + |Dh|^2 + |D^*Dh|^2 \, dV_g^{\ 1/2}, \tag{1.9}$$

where D^* is the L^2 adjoint of D; (D^*D is the so-called rough Laplacian of g). Since D^*D is an elliptic operator, by elliptic regularity, there is a constant $C = C(g)$ such that

$$||h||_{T^{2,2}} \leq ||h||_{L^{2,2}} \leq C(g) \cdot ||h||_{T^{2,2}}. \tag{1.10}$$

Using (1.10), it can be shown that the topology defined by the $T^{2,2}$ metric is the same as the topology defined by the $L^{2,2}$ metric, c.f. again [E,p.21]. However the $L^{2,2}$ and $T^{2,2}$ metrics are far from being uniformly equivalent; the constant C, which by definition is the constant on which one has L^2 elliptic estimates, c.f. [GT, Thm.8.8], cannot be chosen independent of g.

We return to the study of $\mathbb{M}$ in more detail in §7.

§1.2. The solution to the Yamabe problem by Yamabe [Y], Trudinger [Tr] Aubin [Au1] and Schoen [Sc1], implies that in each conformal class $[g] \subset \mathbb{M}_1$ of metrics there is a preferred metric, namely a *Yamabe metric* , i.e. a metric of constant scalar curvature which minimizes the total scalar curvature functional $\mathcal{S}$ (0.1) restricted to the conformal class $[g] \cap \mathbb{M}_1$.

We let $\mathcal{C}$ denote the subset of $\mathbb{M}$ consisting of all Yamabe metrics, and $\mathcal{C}_1$ the subset of unit volume Yamabe metrics; $\mathcal{C}$ is obtained from $\mathcal{C}_1$ by rescaling. If g is a fixed background metric in $[g] \cap \mathbb{M}_1$ and $\bar{g}$ is a Yamabe metric in $[g] \cap \mathbb{M}_1$, then one may write

$$\bar{g} = \psi^4 \cdot g, \tag{1.11}$$

where ψ is a smooth positive function on M, with L^6 norm 1, w.r.t. dV_g, corresponding to $\mathrm{vol}_{\bar{g}} M = \mathrm{vol}_g M$. The equation that $\bar{g}$ have constant scalar curvature $\bar{s} = \inf \mathcal{S}|_{[g]}$ is

$$\psi^5 \cdot \bar{s} = -8\Delta\psi + \psi \cdot s. \tag{1.12}$$

In general, although there always exists at least one Yamabe metric, it is unknown to what extent they are unique in their conformal class. It is well-known, and follows easily from the maximum principle applied to (1.12), that Yamabe metrics $\bar{g} \in \mathcal{C}_1$ are unique in their conformal classes, when the scalar curvature $\bar{s}$ of $\bar{g}$, necessarily constant, is non-positive. In fact, in this case, $\bar{g}$ is the unique metric of constant scalar curvature in $[g] \cap \mathbb{M}_1$. c.f. [Sc2].

Thus, in case $\sigma(M) \leq 0$, any unit volume Yamabe metric has non-positive scalar curvature and so is unique. Further, it is well-known, c.f. [Bes, Cor.4.49], [Sc2] that if $\sigma(M) \leq 0$, any metric of zero scalar curvature is flat, so that, with the exception of flat 3-manifolds, any Yamabe metric on M has negative scalar curvature. Conversely, if M is a flat 3-manifold, then M admits no metrics of positive scalar curvature, c.f. [Sc2], so that $\sigma(M) = 0$. Thus, on a flat 3-manifold, any flat metric realizes the Sigma constant (0.5).

On the other hand, if $\sigma(M) > 0$, i.e. M admits a metric g of positive scalar curvature, then there may be many (necessarily) positive constant scalar curvature metrics in $[g] \cap \mathbb{M}_1$.

We note that in case $\sigma(M) \leq 0$, the space $\mathcal{C}$ of Yamabe metrics is an infinite dimensional submanifold of $\mathbb{M}$. More generally, if $g \in \mathcal{C}$, and $-s_g/2$ is not in the spectrum of the Laplacian, then an $L^{2,2}$ neighborhood of g in $\mathcal{C}$ is an infinite dimensional submanifold of $\mathbb{M}$; c.f. [Bes, Ch.4F], [Ks].

§1.3. As noted in the Introduction, the static vacuum Einstein equations

$$hr = D^2h, \qquad \Delta h = 0, \tag{1.13}$$

will play a fundamental role in the study of the degeneration of Yamabe metrics. Here D^2h is the Hessian of h, i.e. $D\nabla h$, where D is the covariant derivative and ∇ the gradient. The Laplacian will always be defined as $\Delta h = \mathrm{tr}\ D^2h$, (sum of second derivatives), so that Δ has non-positive spectrum.

These are equations for a pair (g, h), defined on an open 3-manifold N. From the maximum principle, it is obvious that there are no (non-trivial) solutions of (1.13) on closed manifolds.

If one considers the 4-manifold $X = N \times S^1$, with warped product metric $g_X = g + h^2 d\theta^2$, then the equations for the Ricci curvature r_X on (X, g_X) are

$$r_X(H,H) = r(H,H) - \frac{1}{h}D^2h(H,H), \qquad r_X(V,V) = -\frac{1}{h} \cdot \Delta h \cdot |V|^2, \tag{1.14}$$

where H is tangent to N, V is tangent to S^1. c.f. [Bes, 9.106]. Thus the static vacuum Einstein equations (1.13) are exactly the equations that the Ricci curvature vanish on (X, g_X).

Using the regularity theory of elliptic equations, c.f. [GT, Ch.8], it is quite standard to prove that $L^{2,2}$ weak solutions of (1.13) are C^∞, in fact real-analytic, in harmonic coordinates, in any domain on which h is bounded away from 0, (where the equations degenerate). Here one uses the fact that, to leading order, the Ricci curvature is given by the Laplacian of the metric in harmonic coordinates. Alternately, one may use regularity estimates for the Einstein metric (1.14) to prove smoothness of weak solutions, c.f. [DK].

Of course, a flat metric, with h an affine function, gives a trivial solution to the static vacuum equations. The canonical solution is given by the Schwarzschild metric (0.17). One has the following remarkable 'black-hole uniqueness theorem', proved by physicists.

Theorem 1.1. [I], [Ro], [BM]. *Let (N, g, h) be a solution to the static vacuum equations (1.13), complete up to the locus $\Sigma = \{h = 0\}$, in the sense that the metric completion $\bar{N}$ of (N, g) is given by $N \cup \Sigma$. Suppose further that the metric g extends smoothly to $N \cup \Sigma$, and that (N, g, h) is* asymptotically flat*, so that, outside a large compact set in $\bar{N}$, the metric g is given in a chart by*

$$g_{ij} = \left(1 + \frac{2m}{t}\right)\delta_{ij} + \gamma_{ij}, \qquad h = 1 - \frac{m}{t} + O(t^{-2}); \tag{1.15}$$

here, $t = |x|$, $|D^k \gamma_{ij}| = O(t^{-2-k})$ and $m > 0$.

If Σ is a compact, possibly disconnected surface, then (N, g, h) is the Schwarzschild metric of mass m for some $m > 0$.

It is shown in [An4] that the assumption that (N, g, h) is asymptotically flat is usually superfluous. Except in some rather rare situations, (which may occur however), it is a consequence of the assumptions that $N \cup \Sigma$ is complete and Σ is compact, (possibly singular).

The locus $\Sigma = \{h = 0\}$ is called the *event horizon* in general relativity, and plays a special role. From the equations (1.13), note that if g is smooth up to Σ, then it follows immediately that $D^2 h = 0$ on Σ. In particular, $|\nabla h| = const.$ on each component of Σ, and it its straightforward to prove that $|\nabla h| > 0$ everywhere on Σ, c.f. [An4, Rmk.1.5]. Hence, each component of Σ is a totally geodesic surface, with $|\nabla h| = const. > 0$.

There are however many other solutions to the static vacuum equations (1.13). Usually, these are either singular at the event horizon Σ, or not complete away from Σ, or both. As indicated above the equations (1.13) may be viewed as an elliptic system, (in harmonic coordinates for example), away from Σ, but the equations formally degenerate at Σ.

We point out though that any weak $L^{2,2}$ static vacuum solution on a domain D, with $L^{2,2}$ potential function h, is smooth, in fact real-analytic, in the interior of D, without any assumptions that h is bounded away from 0 in D. Hence, in this case, $\Sigma \cap D$ is a smooth, totally geodesic surface, and g is smooth up to Σ, as above. Since no use will be made of this sharpening of the local regularity mentioned prior to Theorem 1.1, we do not include a proof, except to note that it is a straightforward exercise in elliptic regularity techniques, on the associated Ricci-flat 4-manifold.

A large and interesting class of explicit solutions of the static vacuum equations are given by the *Weyl solutions*, where (N, g) itself is a warped product of the form

$$N = V \times_f S^1, \qquad g = g_V + f^2 d\theta^2,$$

and (V, g_V) is a Riemannian surface, c.f. [EK, §2.3-9], [Kr, Ch.16-18]; these metrics are discussed in much more detail in [An4]. For metrics of this form, the function

$$r = f \cdot h$$

is harmonic on (V, g_V). Let z be the (locally defined) harmonic conjugate of r on V. Then the function

$$\nu = \log h \tag{1.16}$$

is an axially symmetric harmonic function on a domain in $\mathbb{R}^3$, where $\mathbb{R}^3$ is given cylindrical coordinates $(r, z, \theta), \theta \in [0, 2\pi]$. The set $I = \{\nu = -\infty\}$, (considered as $\cap_n \nu^{-1}(-\infty, \mathrm{n})$), is usually assumed to be non-empty, (c.f. Theorem 3.2(I)), and in most circumstances, (but not always), is a subset of the axis $A = \{r = 0\}$. Note that the event horizon Σ corresponds to the locus I. The metric g is given in these cylindrical coordinates by

$$g = h^{-2}(e^{2\lambda}(dr^2 + dz^2) + r^2 d\theta^2), \tag{1.17}$$

where λ is a solution to the integrability equations

$$\lambda_r = r(\nu_r^2 - \nu_z^2), \qquad \lambda_z = 2r\nu_r\nu_z.$$

Conversely, given any axially symmetric harmonic function ν on a domain in $\mathbb{R}^3$, the equations above determine λ (locally) up to a constant and the metric (1.17) gives a solution to the static vacuum equations with S^1 symmetry.

Thus, Weyl solutions are completely determined locally by an axially symmetric harmonic function ν on a maximal domain D in $\mathbb{R}^3$.

There is a large variety of behaviors in the global geometry of Weyl solutions, c.f. [An4]. Probably the most interesting class of metrics are those for which the potential ν in (1.16) is a globally subharmonic function on $\mathbb{R}^3$. For these, one may use the value distribution theory of subharmonic functions on domains in $\mathbb{R}^3$ to analyse the geometry of the associated Weyl metric.

If ν is subharmonic on $\mathbb{R}^3$ and bounded above, say $\sup \nu = 0$, then the Riesz representation theorem c.f. [Ha, Thms. 3.9, 3.20], implies that ν may be globally represented as

$$\nu(x) = -\int_{\mathbb{R}^3} \frac{1}{|x - \xi|} d\mu_\xi, \tag{1.18}$$

where $d\mu_\xi$ is a positive Radon measure on $\mathbb{R}^3$, the *Riesz measure* of ν. For such axi-symmetric functions, one has the characterization

$$\mathrm{supp}\ d\mu = \bar{I} \Leftrightarrow \mathrm{supp}\ d\mu \subset A.$$

The Riesz measure $d\mu$ in (1.18) encodes all the geometric information on the structure of such Weyl solutions. The fact that the Riesz measure is positive implies that such solutions have positive mass, in the sense of general relativity. Note that $\nu \to \sup \nu = 0$ as the distance to the event horizon goes to ∞.

On the other hand, there are solutions with negative mass. One may just take the function $-\nu$ in (1.18) for example, so that the potential is superharmonic. Note that in this case, u is unbounded *above* on $\mathrm{supp}\, d\mu$, while u tends to its infimum at large distances to supp $d\mu$. Of course in this case, there is no event horizon.

It is worthwhile to list a few standard examples of Weyl solutions and their corresponding measures. In these examples, the metric is given globally by (1.17), and the mass is assumed positive.

From the point of view of the Riesz measure, perhaps the simplest example is the measure $d\mu$ given by a multiple of the Dirac measure at some point on A, so that $\nu = -m/t,\ t(x) = |x|$, is a multiple of the Green's function on $\mathbb{R}^3$. This gives rise to the Curzon solution [Kr, (18.4)],

$$g_c = e^{2m/t}(e^{-m^2r^2/t^4}(dr^2 + dz^2) + r^2 d\theta^2).$$

The event horizon Σ corresponds formally to $\{0\}$ and $u = e^{-m/t}$. This metric has a complicated singularity at the origin.

The Schwarzschild metric (0.17) is a Weyl metric, with measure $d\mu = \frac{1}{2}dA$, where dA is the standard Lebesgue measure on a finite interval, say $[-m, m]$ in A. This gives ν of the form

$$\nu = \frac{1}{2}\ln\frac{r_+ + r_- - 2m}{r_+ + r_- + 2m}, \quad \text{where} \quad r_\pm^2 = r^2 + (z \pm m)^2.$$

As mentioned before, the event horizon Σ here is a smooth totally geodesic 2-sphere of radius $2m$.

It is easy to see that a Weyl solution (N, g) generated by a potential ν as in (1.18) for which supp $d\mu = \bar{I}$ is a compact subset of the axis A, is asymptotically flat, in the sense of (1.15). Further, the simplest or most natural surfaces enclosing any finite number of compact components of $\bar{I}$, and intersecting A outside $\bar{I}$, are 2-spheres in N. Of course if supp $d\mu \subset A$ is non-compact, then the solution cannot be asymptotically flat.

Note that (sub)-harmonic functions of the form (1.18) form a convex cone. In particular, one thus has a natural linear superposition principle for Weyl solutions. For example, one may choose the measure $d\mu = \frac{1}{2}dA$ on two, or any number of disjoint intervals $\{I_j\}$ on the axis A, provided the integral in (1.18) is finite. These correspond to solutions with 'multiple black holes'. Although such solutions are essentially smooth up to the axis A, they do not define smooth Weyl solutions (N, g). There are cone singularities, (called struts in the physics literature), along geodesics (corresponding to $A \setminus \{I_j\}$) joining the 2-spheres of Σ, so that the metric g is not locally Euclidean along such curves. Nevertheless, the curvature of such metrics is uniformly bounded everywhere. Of course, the black hole uniqueness theorem, Theorem 1.1, also implies that such solutions cannot be smooth everywhere, when the number of intervals is finite.

It seems to be unknown whether there are any smooth Schwarzschild type metrics with infinitely many black holes, although it is natural to conjecture that such solutions do not exist, c.f. [An4].

On the other hand, there are Weyl solutions, c.f. [Sz], which are smooth and complete everywhere away from the event horizon Σ, for which Σ, (corresponding to $I \subset$ A), consists of any number, including infinity, of components; here the Weyl metric is highly singular at the event horizon.

It is often useful in analysing the behavior of static vacuum solutions to consider the conformally equivalent metric

$$\tilde{g} = u^2 \cdot g \tag{1.19}$$

on N, c.f. [EK, §2-3.5] and compare with (1.17). An easy calculation, c.f. [Bes, p.59] shows that the Ricci curvature $\tilde{r}$ of $\tilde{g}$ is given by

$$\tilde{r} = 2(d\log u)^2 \geq 0, \tag{1.20}$$

and

$$\widetilde{\Delta} \log u = 0. \tag{1.21}$$

This allows one to use methods and results on spaces with non-negative Ricci curvature and the behavior of harmonic functions on such spaces. Note that the metric $\tilde{g}$ is necessarily singular at the event horizon, even if the event horizon Σ is smooth in (N, g).

§1.4. We briefly summarize some of the main aspects of the theory of *convergence/degeneration* of metrics under uniform curvature bounds, but refer to the primary sources for further details.

Let V_i be a sequence of (possibly open) manifolds and let $\{\gamma_i\}$ be a sequence of smooth Riemannian metrics on V_i. The sequence (V_i, γ_i) is said to converge to (V, γ) in the $C^{1,\alpha}$ topology, if first V_i is diffeomorphic to V, for all i sufficiently large, and second there exist diffeomorphisms $\phi_i : V \to V_i$, such that the pull-back metrics $\phi_i^* \gamma_i$ converge to the metric γ in the $C^{1,\alpha}$ topology on V. This means that there is a smooth coordinate atlas on V for which the component functions of $\{\phi_i^* \gamma_i\}$ converge to the component functions of γ; here the convergence is with respect to the usual $C^{1,\alpha}$ topology for functions defined on domains in $\mathbb{R}^3$. Note that this notion is well-defined for metrics γ which are not necessarily C^∞; for instance, it may well be that the limit metric γ is only a $C^{1,\alpha}$ metric on V.

In exactly the same way one defines convergence with respect to other topologies, for instance the weak or strong $L^{2,2}$ topology. It follows from the Sobolev embedding theorem (1.7) that convergence in the strong $L^{2,2}$ topology implies convergence in the $L^{1,6}$ and $C^{1/2}$ topologies. Further, convergence in the weak $L^{2,2}$ topology implies convergence in the $L^{1,p}$ and C^α topologies, for $p < 6$ and $\alpha < \frac{1}{2}$.

We require the following definitions for a Riemannian 3-manifold (M, g), c.f. [An2, §3].

Definition 1.2. (I) The *μ-volume radius* at x is given by

$$\nu(x) = \sup \left\{ r : \frac{\mathrm{vol}(B_y(s) \cap B_x(r))}{s^3} \geq \mu\omega_3, \text{ for all } y \in B_x(r),\ s \leq r, \right\} \tag{1.22}$$

where ω_3 is the volume of the unit ball in $\mathbb{R}^3$. The parameter μ is chosen to be an arbitrary but fixed small number, e.g. 10^{-2}, so that we will suppress the parameter μ.

We note that trivially one has the bound

$$\nu(x) \leq \big((\mu\omega_3)^{-1} \cdot \mathrm{vol}\, M \big)^{1/3}. \tag{1.23}$$

(II). The $L^{2,2}$ *harmonic radius* $r_h(x)$, at x is the radius of the largest geodesic ball about x in which there exist harmonic coordinates $u_i : B_x(r_h(x)) \to \mathbb{R}$, with respect to which the metric components g_{ij} satisfy

$$e^{-C}\delta_{ij} \leq g_{ij} \leq e^{C}\delta_{ij}, \qquad \text{as bilinear forms,} \tag{1.24}$$

and

$$r_h^{1/2} ||\partial^2 g_{ij}||_{L^2(B_x(r_h(x)))} \leq C. \tag{1.25}$$

(III). The L^2 *curvature radius* $\rho(x)$ is the radius of the largest geodesic ball at x such that for $y \in B_x(\rho(x))$, and $D_y(s) = B_x(\rho(x)) \cap B_y(s)$, one has the bound

$$\frac{s^4}{\operatorname{vol} D_y(s)} \int_{D_y(s)} |r|^2 \leq c_o, \tag{1.26}$$

for any $s \leq \rho(x)$. Note that the left-hand side of (1.26) is not necessarily a monotonic function of s, for a fixed y, thus the need to vary the center point, as in (1.22). Define the harmonic radius r_h, or $r_h(N)$ of (N, g) by $r_h(N) = \inf_{x \in N} r_h(x)$, and similarly for the L^2 curvature radius $\rho(N)$ of N. If N is a complete flat manifold, note that $\rho(N) = \infty$.

The constant C in (1.24)–(1.25) is an arbitrary but fixed parameter that may be taken to be 1. Similarly, c_o is a free parameter, but will be chosen to be a fixed sufficiently small number, say 10^{-3}, throughout the paper. Note that the bounds (1.24)-(1.26) are invariant under rescaling of the metric (one also rescales the coordinate functions u_i). In particular, these radii and the volume radius scale, i.e. behave under rescalings of the metric, as distance functions do. Observe from the definition that if $y \in B_x(\rho(x))$, then

$$\rho(y) \geq \operatorname{dist}(y, \partial B_x(\rho(x))). \tag{1.27}$$

The same estimate holds for r_h. Thus, ρ and r_h are Lipschitz functions, with Lipschitz constant 1. By the Sobolev embedding theorem, $L^{2,2} \subset C^{1/2}$, so that one also has $C^{1/2}$ control of the metric components on $B_x(r_h(x))$.

It is important to note that the $L^{2,2}$ harmonic radius and the L^2 curvature radius are continuous with respect to convergence in the strong $L^{2,2}$ topology. Thus, if $(M_i, g_i) \to (M, g)$ in the $L^{2,2}$ topology, with $x_i \to x$, then

$$\lim_{i \to \infty} r_h(x_i, g_i) = r_h(x, g), \quad \text{and} \quad \lim_{i \to \infty} \rho_{g_i}(x_i) = \rho_g(x). \tag{1.28}$$

This is more or less obvious for ρ and is proved for r_h in [An3, Lemma 2.2], c.f. also [AC,Prop.1.1]. However, (1.28) is not true if the convergence is only in the weak $L^{2,2}$ topology.

There is an obvious relation between r_h and ρ, namely

$$\rho(x) \geq c \cdot r_h(x), \tag{1.29}$$

where $c = c(C, c_o)$. To see this, we may assume (by rescaling) that $r_h(x) = 1$. The bound (1.24) then implies upper and lower bounds on $\operatorname{vol} B_x(1)$, while the bound (1.25) provides a bound on the L^2 norm of curvature on $B_x(1)$. This shows $\rho(x) \geq c$, as required.

On the other hand, the opposite inequality, $\rho(x) \leq c \cdot r_h(x)$, is not true, as seen for example on compact flat manifolds, or more generally on manifolds which are highly collapsed on the scale of their curvature, that is for which $\nu << \rho$. On the other hand, under the presence of a lower bound on $\nu(x)$, one does obtain a bound of the form $\rho(x) \leq c \cdot r_h(x)$, provided $\rho(x)$ is not too large, i.e. $\rho(x) \leq K$, where $c = c(\nu(x), K)$, c.f. [An2, §3],[An3].

The natural bounds on sequences of Yamabe metrics that we obtain are bounds on the L^2 norm of (components of) the curvature. The following results, proved in [An2, §3], summarize the behavior of sequences of metrics on 3-manifolds, having a uniform L^2 bound on curvature. These are extensions of the fundamental (L^∞) Cheeger-Gromov theory of convergence and collapse of Riemannian manifolds, see [C], [G], [CG1,2].

Theorem 1.3. *Let $\{g_i\}$ be a sequence of metrics in $\mathbb{M}_1(M)$, where M is a closed 3-manifold. Suppose there is a uniform bound*

$$\int_M |r_{g_i}|^2 dV_i \leq \Lambda < \infty. \tag{1.30}$$

Then there is a subsequence, also called $\{g_i\}$, and diffeomorphisms ψ_i of M such that exactly one of the following occurs:

(I) *(Convergence) The metrics $\psi_i^* g_i$ converge in the weak $L^{2,2}$ topology to a $L^{2,2}$ metric g_o on M.*

(II) *(Collapse) The metrics $\psi_i^* g_i$ collapse M along a sequence of orbit structures of a graph manifold structure. In particular, M is necessarily a graph manifold. The metrics $\psi_i^* g_i$ collapse each orbit $\mathcal{O}_x$, (namely a circle or torus), of a sequence of orbit structures to a point, as $i \to \infty$, i.e.* $\mathrm{diam}_{\psi_i^* g_i} \mathcal{O}_x \to 0$, $\forall x \in M$.

(III) *(Cusps) There is a maximal open domain Ω, (not necessarily connected), such that the metrics $\psi_i^* g_i$ converge, uniformly on compact subsets in the weak $L^{2,2}$ topology, to an $L^{2,2}$ metric g_o defined on Ω, of volume ≤ 1. Any smooth compact domain $K \subset \Omega$ embeds in M and for sufficiently large $K \subset \Omega$, the complement $M \setminus K$ has the structure of a graph manifold, part of which is collapsed along a sequence of orbit structures as in II.*

Graph manifolds are 3-manifolds which are unions of Seifert fibered spaces glued along toral boundary components; thus, they have naturally defined embedded circles or tori, see [CG1], [An1,2]. From the Sobolev embedding theorem, the convergence in cases I and III above is also in the $L^{1,p}$ and C^α topologies, $p < 6$, $\alpha < \frac{1}{2}$.

The distinction between these three cases is determined by the behavior of the volume radius ν_i. Case I occurs if ν_i is uniformly bounded below, $\nu_i \geq \nu_o \geq 0$, Case II occurs if $\nu_i \to 0$ everywhere on M, while in Case III, there are regions in (M, g_i) where ν_i is bounded below and regions where it goes to 0.

The following elementary result shows that $\{(M, g_i)\}$ degenerates in the sense of (0.8) only if $\rho_i(x_i) \to 0$, for some $x_i \in M$.

Lemma 1.4. *Let g be a unit volume metric on M for which there is a uniform lower bound on the L^2 curvature radius $\rho(M)$, say*

$$\rho(M) \geq \rho_o. \tag{1.31}$$

Then there is an explicit constant $\Lambda = \Lambda(\rho_o)$ such that

$$\int_M |z|^2 dV \leq \Lambda. \tag{1.32}$$

Proof. From the definition of ρ, we have

$$\int_{B_x(\rho_o)} |z|^2 dV \leq c_o \frac{\mathrm{vol}\, B_x(\rho_o)}{\rho_o^4},$$

for any geodesic ball $B_x(\rho_o) \subset (M, g)$. Choose a maximal family of disjoint balls $B_{x_k}, k = 1, ..., m$ of radius $\rho_o/4$ in (M, g); thus the corresponding balls of radius $\rho_o/2$ cover M. Now observe that there is a uniform upper bound N on the multiplicity of this covering, independent of ρ_o, M. In fact, since c_o in (1.26) is sufficiently small, ($c_o = 10^{-3}$ suffices), the geometry of $B(\rho_o/2)$ is very

close to that of a Euclidean ball, (or its quotient by Euclidean isometries), so that the Besicovitch covering theorem holds, c.f. [M, Theorem 2.7]. Thus

$$\int_M |z|^2 dV \leq N \sum \int_{B_{x_k}(\frac{\rho_o}{2})} |z|^2 dV \leq \frac{Nc_o}{\rho_o^4} \sum \mathrm{vol}\, B_{x_k}\left(\frac{\rho_o}{2}\right) \leq C\frac{Nc_o}{\rho_o^4} \sum \mathrm{vol}\, B_{x_k}\left(\frac{\rho_o}{4}\right) \leq C\frac{Nc_o}{\rho_o^4}\, \mathrm{vol}\, M.$$

■

There are also local versions of Theorem 1.3 which will be frequently used below. While there are related results which hold in the collapse case, c.f. [An2, §3], we consider only the non-collapsing situation here; c.f. [An2, Remark 3.6] for the following result.

Theorem 1.5. *Let (U_i, g_i, x_i) be a pointed sequence of smooth Riemannian 3-manifolds such that*

$$\rho_i(x_i) \geq \rho_o > 0, \qquad \nu_i(x_i) \geq \nu_o > 0, \qquad \mathrm{diam}\, U_i \leq D < \infty,$$

and

$$\mathrm{dist}(x_i, \partial U_i) \geq \delta,$$

for some arbitrary positive constants ρ_o, ν_o, D, δ. Then for any given $\varepsilon > 0$ sufficiently small, there are smooth domains $V_i \subset U_i$ with $\varepsilon/2 \leq \mathrm{dist}(\partial V_i, \partial U_i) \leq \varepsilon$, such that a subsequence of (V_i, g_i) converges, modulo diffeomorphisms, to a limit $L^{2,2}$ Riemannian manifold (V, g_o). The convergence is in the weak $L^{2,2}$ topology. In particular, the limit domain V embeds in the domains U_i.

In studying the degeneration of a sequence $\{g_i\}$ of Yamabe metrics on M, we will *blow-up* the metrics in neighborhoods of points where the curvature of g_i goes to infinity, i.e. consider the behavior of the rescaled sequence

$$g_i' = \rho(x_i)^{-2} \cdot g_i, \tag{1.33}$$

when $\rho(x_i) \to 0$; of course, $\rho(x_i)$ is the L^2 curvature radius of g_i at x_i. Theorem 1.5 will be used to examine the behavior of this sequence. In considering limits of $\{g_i'\}$, one must always consider based limits, i.e. limits w.r.t. a sequence of base points; the points $\{x_i\}$ will always be chosen to be the base points. Thus, assuming for instance that $\nu'(x) \geq \nu_o$, for all $x \in B_{x_i}'(1)$, for some $\nu_o \geq 0$, the pointed sequence $\{(B_{x_i}'(1), g_i', x_i)\}$ has a subsequence converging weakly in $L^{2,2}$ to a limit (B', g', x), $\mathrm{x} = \lim x_i$, with limit $L^{2,2}$ metric g'. The convergence $g_i' \to g'$ is also understood to be in the pointed Gromov-Hausdorff topology [G, Ch.5A].

Any smooth compact subdomain of B' is naturally, but not canonically, embedded as a (very small) domain in (M, g_i) and the structure of the limit (B', g') mirrors the very small scale behavior of (M, g_i) near x_i as $i \to \infty$.

Finally, some general remarks. Since we are constantly dealing with the behavior of sequences $\{g_i\}, \{g_i'\}$ etc., we will often drop the subscript i, or prime, in order to simplify notation, when there is no danger of confusion. The main point is to establish uniform estimates, independent of i. Similarly, we will often pass to subsequences to obtain convergence, without always indicating the specific subsequence. A sequence $\{\alpha_i\}$ is said to sub-converge if a subsequence converges.

2. Initial Global Estimates for Yamabe Metrics

In this section, we derive a number of simple global relations on the behavior of the gradient of the (normalized) scalar curvature functional, i.e. the traceless Ricci curvature $-z$, as well various components of z, on the space of Yamabe metrics $\mathcal{C}$.

For a given smooth metric $g \in \mathcal{C}$, consider the operator $L : S^2(M) \to C^\infty(M)$ giving the derivative or linearization of the scalar curvature function at g, i.e.

$$L(\alpha) = s'(\alpha) = \frac{d}{dt} s_{(g+t\alpha)}. \tag{2.1}$$

This operator has been classically studied, for instance in general relativity, and is given, c.f. [Bes, Ch.1K], by

$$L(\alpha) = -\Delta \operatorname{tr} \alpha + \delta\delta\alpha \ - \langle r, \alpha \rangle, \tag{2.2}$$

The L^2 adjoint of L, (w.r.t. the metric g), $L^* : C^\infty(M) \to S^2(M)$, is the expression

$$L^*(h) = D^2 h - \Delta h \cdot g - h \cdot r. \tag{2.3}$$

This is an (overdetermined) elliptic operator, and thus by general elliptic theory there is a splitting of $T_g\mathbb{M}$, orthogonal w.r.t. the L^2 metric, of the form

$$T_g\mathbb{M} = \operatorname{Im} L^* \oplus \operatorname{Ker} L. \tag{2.4}$$

To our knowledge, the splitting (2.4) first appeared in work of Berger-Ebin [BE], where it is attributed to Fadeev and Nirenberg, c.f. also [Bes, Ch.4F].

Of course one sees immediately that

$$- r = L^*(1), \tag{2.5}$$

corresponding to the fact that the L^2 gradient of the functional $v^{-1} \int s \, dV$, (at a Yamabe metric), is given by $-r$.

On the other hand, one may decompose the traceless Ricci tensor z $(= z_g)$ with respect to this splitting and write

$$z = L^* f + \xi, \tag{2.6}$$

where $\xi \in \operatorname{Ker} L$ and f is a smooth function on M. Clearly ξ and $L^* f$ are uniquely determined by z. If

$$\operatorname{Ker} L^* = \{0\}, \tag{2.7}$$

then f is also uniquely determined by z.

To examine Ker L^*, suppose $h \in \operatorname{Ker} L^*$, so that $D^2 h - \Delta h \cdot g - h \cdot r = 0$. Taking the trace, one obtains

$$- 2\Delta h - sh = 0, \tag{2.8}$$

i.e. h is an eigenfunction of the Laplacian, with eigenvalue $-\frac{s}{2}$. Suppose first that $\sigma(M) \le 0$. Then it follows, see §1.2, that $s < 0$, (unless M is a flat 3-manifold), and since the Laplacian has non-positive spectrum, the only solution of (2.8) is $h = 0$. In other words, $\sigma(M) \le 0$ implies Ker $L^* = 0$, except in the special case that (M, g) is a flat 3-manifold, where Ker L^* consists of the

constant functions. In case $\sigma(M) > 0$, Ker L^* may well be non-zero; see §3.6 and §6.4 for further discussion.

Remark 2.1. It is worth emphasizing at this point that although z is locally determined by g, neither L^*f nor ξ in (2.6) are locally determined by g, (in contrast to the terms in (2.5)). They both depend on the global geometry of (M, g), in particular on the global volume or scale of (M, g). Thus they cannot be expressed locally in terms of the metric and its derivatives; see Theorem 2.10 for a simple concrete illustration of this.

On the other hand, in analogy to (2.5), note that $z = H^*(-1)$, where $H^*(f) = D^2 f - \Delta f \cdot g - f \cdot z$ is the L^2 adjoint of the operator H, $H(\alpha) = -\Delta \operatorname{tr} \alpha + \delta\delta\alpha - \langle z, \alpha\rangle$, giving the derivative or linearization of the functional $v^{2/3} \cdot s$ on $\mathcal{C}$.

Returning to the general discussion, we may add the equations (2.5) and (2.6); this gives the L^2 splitting for the metric g, (considered as an element of $T_g\mathbb{M}$), i.e.

$$-\frac{s}{3} \cdot g = L^*(1+f) + \xi, \tag{2.9}$$

so that, setting $u = 1 + f$ gives

$$L^*(u) + \xi = -\frac{s}{3} \cdot g. \tag{2.10}$$

Note that at least in the case $\sigma(M) \leq 0$, g is never in Im L^* unless (M, g) is Einstein. To see this, suppose there is an h such that

$$L^* h = g.$$

Taking the trace gives

$$2\Delta h + sh = -3.$$

It follows that $h + \frac{3}{s}$ is an eigenfunction of Δ, with eigenvalue $-\frac{s}{2}$, so that the argument above implies (M, g) must be Einstein. The same reasoning shows that if $\sigma(M) \leq 0$, then $z \in$ Im L^* only if (M, g) is Einstein; this is also conjectured to be the case when $\sigma(M) > 0$, c.f. [Bes, Remark 4.48].

Thus, the pair (f, ξ) measure in a certain global way the deviation from g being an Einstein metric on M.

Taking the L^2 norm of both sides of (2.10) gives the following simple but important estimate.

Theorem 2.2. *For f and ξ defined as above, and $u = 1 + f$, $v = \operatorname{vol} M$, one has*

$$\int |L^*u|^2 + \int |\xi|^2 = \frac{s^2}{3} \cdot v. \tag{2.11}$$

■

Thus, one has apriori bounds on the L^2 norms of $L^*(u)$ and ξ. Referring back to (2.6), it follows that the component of z in Ker L is apriori bounded in L^2. Thus, z is uncontrolled in L^2, say on a sequence $g_i \in \mathcal{C}_1$ with scalar curvature bounded below, only in the direction Im L^*.

The splittings (2.6), (2.10) immediately give corresponding trace equations

$$2\Delta f + sf = \operatorname{tr} \xi, \tag{2.12}$$

$$2\Delta u + su = \operatorname{tr} \xi + s. \tag{2.13}$$

These equations easily lead to the following identities.

Lemma 2.3. *For an arbitrary Yamabe metric $g \in \mathcal{C}$, one has the relations*

$$\int |\xi|^2 = -\frac{s}{3} \int \mathrm{tr}\ \xi, \tag{2.14}$$

and

$$\int \mathrm{tr}\ \xi = s \int f. \tag{2.15}$$

Proof. For $\xi \in \mathrm{Ker}\ L$, note by (2.2) that $\int \langle r, \xi \rangle = 0$. Hence

$$\int \langle z, \xi \rangle = \int \langle r, \xi \rangle - \frac{s}{3} \int \langle g, \xi \rangle = -\frac{s}{3} \int \mathrm{tr}\ \xi.$$

But also

$$\int \langle z, \xi \rangle = \int \langle L^* f, \xi \rangle + \int |\xi|^2 = \int |\xi|^2.$$

This gives (2.14).

The equation (2.15) follows immediately from the trace equation (2.12) by integrating over M. ∎

Of course (2.14) and (2.15) combine to give

$$\int |\xi|^2 = -\frac{s^2}{3} \int f. \tag{2.16}$$

Note that if $g \in \mathcal{C}_1$, then (2.11) implies that the L^2 norm of ξ is bounded by $\frac{s^2}{3}$.

Definition 2.4. For $g \in \mathcal{C} \cap \mathbb{M}_1$, define the quantities $\delta \in [0,\ 1]$ and $\lambda \in [0,\ 1]$ by

$$\delta = -\int_M f, \qquad \lambda = 1 - \delta = \int_M u. \tag{2.17}$$

The behavior of the quantities δ and λ plays a fundamental role in the discussion. For $g \in \mathcal{C}$ not of unit volume, δ, λ are defined by the averages in (2.17); thus, they are invariant under scaling.

One immediate consequence of this discussion is obtained by taking the L^2 inner product of (2.10) with z, to give the interesting bound

$$\int u|z|^2 = \int \langle \xi, z \rangle = \int |\xi|^2 \leq \frac{s^2}{3} \cdot v. \tag{2.18}$$

In particular, if one had an estimate of the form $u \geq u_o > 0$ on (M, g), then (2.18) gives an L^2 bound on z; compare with §0.

Another important L^2 orthogonal splitting of $T_g\mathbb{M}$ relevant to the study of the functional $\mathcal{S}|_{\mathcal{C}}$ is given by

$$T_g\mathbb{M} = T_g\mathcal{C} \oplus N_g\mathcal{C}, \tag{2.19}$$

where

$$T_g\mathcal{C} = \{\chi \in T_g\mathbb{M} : L(\chi) = const.\}, \quad N_g\mathcal{C} = \left\{\tau \in T_g\mathbb{M} : \tau = L^*(h), \int_M h = 0\right\}, \tag{2.20}$$

as in (0.10). At least in the case where $\sigma(M) \leq 0$, i.e. where Yamabe metrics are the unique metrics of constant scalar curvature in their conformal class, the spaces in (2.19) are the tangent and normal spaces to the space of Yamabe metrics $\mathcal{C}$. In fact, using the splitting (2.19), one may show that $\mathcal{C}$ is an infinite dimensional submanifold of $\mathbb{M}$, c.f. [Bes, Ch.4F], [Ks]. In case however $-s/2 \in Spec\,(\Delta)$, the spaces $T_g\mathcal{C}$ and $N_g\mathcal{C}$ should only be considered as formal tangent spaces to $\mathcal{C}$. (Recall that Δ has non-positive spectrum).

Note that the splittings (2.4) and (2.19) differ by only 1-dimensional factors. We may then decompose z also with respect to this splitting, and write

$$z = z^T + z^N. \tag{2.21}$$

The component $-z^T$, more precisely $-v^{-1/3} \cdot z^T$, is the L^2 gradient of the functional

$$v^{2/3} \cdot s : \mathcal{C} \to \mathbb{R}, \tag{2.22}$$

(again assuming $-s/2 \notin Spec\,(\Delta)$). One sees this by observing that $-v^{-1/3} \cdot z$ is the L^2 gradient of the (volume normalized) total scalar curvature functional $\mathcal{S} : \mathbb{M} \to \mathbb{R}$ in (0.1) and z^T is the L^2 projection of z onto $\mathcal{C}$.

Now the space $T\mathcal{C}$ may be further decomposed into

$$T\mathcal{C} = T\mathcal{L} \oplus N\mathcal{L} = \text{Ker } L \oplus (\text{Im } L^* \cap T\mathcal{C}), \tag{2.23}$$

where

$$T\mathcal{L} = \{\xi \in T\mathcal{C} : L(\xi) = 0\}, \qquad N\mathcal{L} = \{\phi \in T\mathcal{C} : \phi = L^*(h), \text{ some } h\}. \tag{2.24}$$

Clearly, the factor $N\mathcal{L} = (\text{Im } L^* \cap T\mathcal{C})$ is 1-dimensional.

Observe that $T\mathcal{L}$ corresponds to the tangent space to the level sets $\mathcal{L}$ of the functional $s = \mathcal{S}|_{\mathcal{C}}$ while $N\mathcal{L}$ corresponds to the normal space of $\mathcal{L}$ in $T\mathcal{C}$. The level sets $\mathcal{L}$ of s and the level sets $\mathcal{H}$ of $v^{2/3} \cdot s$ are obviously hypersurfaces in $\mathcal{C}$ at non-critical points of these functionals. These of course coincide on $\mathcal{C} \cap \mathbb{M}_1$, but do *not* coincide off $\mathbb{M}_1$. Note also that by definition, z^T is L^2 orthogonal to the tangent spaces of $\mathcal{H}$.

The splittings (2.4), (2.19) and (2.23) are all compatible with the splitting (1.2), since constant scalar curvature is a diffeomorphism invariant, see also [BE].

In particular, the L^2 projection of z, $z^T \in T\mathcal{C}$ can be split further as

$$z^T = L^*k + \xi, \qquad \text{where } L^*k \in N\mathcal{L},\ \xi \in T\mathcal{L}. \tag{2.25}$$

Lemma 2.5. *The function k is characterized by the basic property that*

$$LL^*k = const. = -\frac{1}{v}\int |z^T|^2. \tag{2.26}$$

Proof. Apply the operator L to both sides of (2.25) to obtain

$$L(z^T) = LL^*k,$$

since $L(\xi) = 0$. Since $z^T \in T\mathcal{C}, L(z^T)$ is a constant function and

$$\int L(z^T)dV = \int \langle L^*1, z^T\rangle dV = -\int \langle r, z^T\rangle dV =$$

$$= -\int |z^T|^2 dV - \int \langle z^T, z^N\rangle dV - \frac{s}{3}\int \text{tr } z^T dV.$$

The components z^T and z^N are L^2 orthogonal. Also $\int \mathrm{tr}\ z^T$ is the derivative of the functional $\mathcal{S}|_{\mathcal{C}}$ in the direction of the metric g, i.e. its derivative under homothetic changes of the metric. Since the functional is scale-invariant, this derivative is 0, and the result follows.

■

A straightforward calculation shows that the operator LL^* is given by

$$LL^*v = 2\Delta\Delta v + 2s\Delta v - \langle D^2 v, r\rangle + v|r|^2. \tag{2.27}$$

Clearly, when Ker $L^* = \{0\}$, the space of functions ϕ such that $LL^*\phi = c$, for some constant c, is 1-dimensional. This gives the following relation between the functions u and k.

Proposition 2.6. *For a Yamabe metric $g \in \mathcal{C}$, with $s_g < 0$, one has the relation*

$$\frac{u}{\lambda} = -\frac{k}{\delta}. \tag{2.28}$$

In general, the relation (2.28) holds $\mod \mathrm{Ker}\ L^*$.

Proof. We have

$$LL^*u = L\left(-\frac{s}{3}\cdot g - \xi\right) = -\frac{s}{3}L(g) = \frac{s^2}{3}; \tag{2.29}$$

the last equality just corresponds to the fact that varying the scalar curvature s in the direction of g, i.e. by a homothety, just changes s by a constant. Thus, by (2.26) and (2.29), both $LL^*k = \alpha$ and $LL^*u = \beta$ are constant. It follows that

$$LL^*\left(\frac{k}{\alpha} - \frac{u}{\beta}\right) = 0.$$

If Ker $L^* = 0$, we see that

$$\frac{u}{\beta} = \frac{k}{\alpha},$$

so that α and β are the mean values of k and u respectively. From (2.6) and (2.25), we have

$$z^N = L^*(f - k), \tag{2.30}$$

so that, in particular, since $z^N \in N\mathcal{C}$,

$$\int f = \int k = -\delta. \tag{2.31}$$

This implies (2.28).

■

Using (2.29), it is easy to see that the function u minimizes the L^2 norm of Im L^* among functions with the same mean value, i.e.

Proposition 2.7. *The function u/λ, (or k/δ), is characterized uniquely by the fact that*

$$\int |L^*(u/\lambda)|^2 dV \leq \int |L^*\phi|^2 dV, \tag{2.32}$$

for all functions ϕ on (M, g) with mean value 1.

Proof. Write

$$\int |L^*\phi|^2 dV = \int |L^*(\phi - u/\lambda) + L^*(u/\lambda)|^2 dV =$$

$$= \int |L^*(\phi - u/\lambda)|^2 dV + \int |L^*(u/\lambda)|^2 dV - 2\int \langle L^*(\phi - u/\lambda), L^*(u/\lambda)\rangle dV.$$

But

$$\int \langle L^*(\phi - u/\lambda), L^*(u/\lambda)\rangle dV = \int \langle(\phi - u/\lambda), LL^*(u/\lambda)\rangle dV = 0,$$

where the last equality follows from the fact that LL^*u is constant by (2.29), and ϕ and u/λ have the same mean value.

■

The equation (2.28) contains the basic relation between the functions f and k, relating z and z^T. It is useful to derive several further relations.

Lemma 2.8. *The following identities hold for any Yamabe metric (M,g):*

$$\xi + \frac{s}{3}\cdot g = \lambda(z^T + \frac{s}{3}\cdot g), \tag{2.33}$$

$$\int |L^*k|^2 = \frac{s^2}{3}\frac{\delta^2}{\lambda}\cdot v, \tag{2.34}$$

$$\int |z^T|^2 = \frac{s^2}{3}\frac{\delta}{\lambda}\cdot v. \tag{2.35}$$

Proof. First, recall that

$$L^*u = -\xi - \frac{s}{3}\cdot g \qquad \text{and} \qquad L^*k = z^T - \xi.$$

From (2.28), one then obtains

$$\frac{\delta - 1}{\delta} z^T = \frac{\delta - 1}{\delta}\xi - \xi - \frac{s}{3}\cdot g = -\frac{1}{\delta}\xi - \frac{s}{3}\cdot g,$$

so that from Definition 2.4,

$$\xi + \frac{\delta\cdot s}{3}\cdot g = \lambda z^T. \tag{2.36}$$

which implies (2.33). Thus the tensors $\xi + \frac{s}{3}g$ and $z^T + \frac{s}{3}g$ are proportional. If ξ_o and z_o^T denote the trace-free parts of ξ and z^T, i.e. $\xi_o = \xi - \frac{tr\xi}{3}\cdot g$, then one also has from (2.33) that

$$\xi_o = \lambda\cdot z_o^T.$$

Next, from (2.28) and (2.29), we find

$$LL^*k = -\frac{s^2}{3}\frac{\delta}{\lambda}, \tag{2.37}$$

Multiply (2.37) by k and integrate over M, using (2.31) to obtain (2.34). Finally from (2.16), we have $\int |\xi|^2 = \frac{s^2}{3}\delta\cdot v$, so that (2.25) and (2.34) give (2.35).

■

We now turn to the trace equation (2.12) in order to obtain estimates on f, or u.

Proposition 2.9. *For any unit volume Yamabe metric $g \in \mathcal{C}_1$, with $s_g < 0$, one has the estimate*

$$||f||_{T^{2,2}} \leq C, \tag{2.38}$$

where C is a constant depending only on $|s_g|$.

Proof. Recall from (1.9), that the $T^{2,2}$ norm is given by

$$||f||^2_{T^{2,2}} = ||f||^2_{L^2} + ||df||^2_{L^2} + ||\Delta f||^2_{L^2}.$$

Multiplying (2.12) by f and integrating gives

$$\begin{aligned} 2\int |df|^2 - s\int f^2 &= -\int f \cdot \mathrm{tr}\ \xi \\ &\leq \tfrac{1}{2}|s|\int f^2 + \tfrac{1}{2}|s|^{-1}\int (\mathrm{tr}\ \xi)^2, \end{aligned} \tag{2.39}$$

so that, since $s < 0$,

$$2\int |df|^2 + \tfrac{1}{2}|s|\int f^2 \leq \tfrac{1}{2}|s|^{-1}\int (\mathrm{tr}\ \xi)^2 \leq \tfrac{1}{2}|s|\delta, \tag{2.40}$$

where the last inequality follows from (2.16). Thus, one has an apriori bound

$$||f||^2_{L^{1,2}} \leq C = C(|s|).$$

Returning to the trace equation (2.12), it also then follows that

$$||\Delta f||^2_{L^2} \leq C = C(|s|).$$

∎

We note that the estimate (2.38) is false in case $s_g > 0$, see §3.6 for further discussion, while for $s_g = 0$, it is borderline. Namely if g is a Yamabe metric with $s_g = 0$, then either $\sigma(M) > 0$ or $\sigma(M) = 0$. In the former case, it is obvious that $f \equiv -1$ and $\xi = 0$ is one solution of the z-splitting equation (2.6) and hence it is the only solution since Ker $L^* = 0$ when $s_g = 0$. Thus, in this case we have $u \equiv 0$. The splittings (2.6), (2.10) contain no essential information in this situation; they are equivalent to the identity (2.5). If however $\sigma(M) = 0$, then g realizes the Sigma constant on M, and thus g is Einstein, with $r = 0$. Thus, the solution to (2.6) is given by $f = const.$ and $\xi = 0$, where the constant for f is however undetermined.

The bound (2.38) implies that there is an apriori bound for f in the $T^{2,2}$ norm, in case $\sigma(M) \leq 0$. On the other hand, we will see later in §3 that it is far from true that one has uniform bounds for $||D^2 f||_{L^2}$, i.e. uniform bounds for the $L^{2,2}$ norm of f. This will be the case when one has non-flat blow-up limits. Thus, one has a breakdown of (uniform) elliptic regularity on $\{g_i\}$, c.f. (1.10).

We refer to §4.2 for some further discussion on the L^2 behavior of u or f.

Next, consider the L^2 bound on $z^T = L^*k + \xi$. From Theorem 2.2, ξ is uniformly controlled in L^2, while k is a function in the 1-dimensional space Im $L^* \cap T\mathcal{C}$. Thus, one would expect to be able to control z^T in a natural way. This is given by the following general result, which illustrates in a simple manner the highly global nature of the L^2 projection operator onto $T\mathcal{C}$.

Theorem 2.10. *Let g be a unit volume Yamabe metric on M. Suppose there is a point $x \in M$, and arbitrary but fixed constants ρ_o, $\nu_o \geq 0$ such that*

$$\rho(x) \geq \rho_o, \qquad \nu(x) \geq \nu_o. \tag{2.41}$$

Then there is a constant $K = K(\rho_o, \nu_o)$ such that

$$\int_M |z^T|^2 dV_g \leq K. \tag{2.42}$$

Proof. Since $z^T \in T_g\mathcal{C}$ is the L^2 orthogonal projection of z onto $T\mathcal{C}$, we have

$$\int_M |z^T|^2 dV_g = \inf_{\int \phi = 0} \int_M |z - L^*\phi|^2 dV_g, \tag{2.43}$$

where the infimum is over all smooth functions ϕ on M with 0 mean value on (M, g); recall $N\mathcal{C} = \mathrm{Im}\ L^*\phi$, over mean value 0 functions on (M, g).

A straightforward computation gives

$$\int_M |z - L^*\phi|^2 dV_g = \int_M \{(1+\phi)^2|z|^2 + |D^2\phi|^2 + (\Delta\phi)^2 - \frac{2}{3}s|d\phi|^2 + 2z(d\phi, d\phi) + \frac{s^2}{3}\phi^2\} dV_g \tag{2.44}$$

$$= \int_M \{(1+\phi)^2|z|^2 - |D^2\phi|^2 + 3(\Delta\phi)^2 - \frac{4}{3}s|d\phi|^2 + \frac{s^2}{3}\phi^2\} dV_g,$$

where the last inequality follows from the use of the Bochner-Lichnerowicz formula to eliminate the term $z(d\phi, d\phi)$.

Given $x \in M$ satisfying (2.41), choose a function ϕ, with $\phi \geq -1$ everywhere on M, $\phi \equiv -1$ on $M \setminus B$, where $B = B_x(\rho_o/2)$ and with 0 mean value on (M, g). Note that since $\mathrm{vol}\, B \geq c \cdot \min((\rho_o, \nu_o))^3$, so that B has a definite proportion of the volume of (M, g), one may choose such a ϕ so that $\sup\ \phi \leq H$, where H depends only on ρ_o, ν_o. Now in the ball B the geometry of g is controlled in the $L^{2,2}$ topology, see the discussion concerning (1.29). It follows that one may choose ϕ so that the $L^{2,2}$ norm of ϕ is also uniformly controlled in B.

For such a choice of ϕ, it is clear that the expression on the right in (2.44) is uniformly controlled by ρ_o and ν_o. The estimate (2.42) then follows immediately from (2.44).

∎

Remark 2.11. In a related vein, suppose $\{g_i\}$ is a sequence of unit volume Yamabe metrics on M, and $\{x_i\}$ is a sequence of points in M satisfying (2.39). Then the local estimate

$$\int_{B_i} |z^T|^2 dV_{g_i} \to 0 \quad \text{as } i \to \infty, \tag{2.45}$$

implies the global estimate

$$\int_M |z^T|^2 dV_{g_i} \to 0 \quad \text{as } i \to \infty, \tag{2.46}$$

where $B_i = B_{x_i}(\rho_o)$. To see this, from (2.25) and (2.26), we have $L(z^T) = -||z^T||_{L^2}$. Multiplying this by a suitable cutoff function supported in B_i implies

$$\mathrm{vol}_{g_i} B_i \cdot ||z^T||_{L^2} \leq c \int \langle L^*, z^T \rangle \leq c \int_{B_i} |z^T|^2,$$

which gives the result.

Remark 2.12. From (2.35), we see that in case $\sigma(M) \neq 0$, a bound on $||z^T||_{L^2}$ is equivalent to a bound on λ away from 0. More precisely, if $\{g_i\}$ is a sequence of unit volume Yamabe metrics with scalar curvature bounded away from 0 (and $-\infty$), then $||z^T||_{L^2}$ remains uniformly bounded exactly when λ, the mean value of u, remains bounded away from 0.

On the other hand, this is certainly not the case when $\sigma(M) = 0$ or more generally when $s_{g_i} \to 0$. Suppose for instance (M, g_o) is a flat 3-manifold, so that g_o realizes $\sigma(M)$. Let g_t be a smooth curve of unit volume Yamabe metrics on M through g_o, and let $\alpha = \frac{dg_t}{dt}$ satisfy $||\alpha||_{L^2(g_t)} = 1$. We have

$$\frac{d}{dt}s_{g_t} = \int_M \langle z^T, \alpha\rangle \ \leq \left(\int_M |z^T|^2\right)^{1/2},$$

so that

$$\left(\frac{d}{dt}s_{g_t}\right)^2 \ \leq \int_M |z^T_{g_t}|^2.$$

Since s_{g_t} is a smooth function of t, with $s_{g_o} = 0$, the maximal value, it is clear that

$$-s_{g_t} << -\frac{d}{dt}s_{g_t},$$

as $t \to 0$. Thus, it follows that

$$\frac{1}{s^2}\int_M |z^T|^2 dV \to \infty \qquad \text{as } t \to 0. \tag{2.47}$$

By (2.35), this implies that

$$\lambda \to 0, \tag{2.48}$$

as $t \to 0$. Further, by the proof of Proposition 2.8, $|\nabla u| \to 0$ in L^2 and since u converges smoothly to its limit here, we have

$$u \to 0 \qquad \text{in } C^o.$$

Note that in this example, one still has of course $||z^T||_{L^2} \leq C$, in fact $||z^T||_{L^2} \to 0$ as $t \to 0$.

3. Existence of Non-Flat Blow-Ups.

In this section, we will analyse the degeneration of sequences of Yamabe metrics using the structural results in §2. We recall (Lemma 1.4), that a sequence of unit volume Yamabe metrics $\{g_i\}$ degenerates, in the sense that the curvature becomes unbounded in L^2, only if the L^2 curvature radius $\rho_i(M)$ of (M, g_i) goes to 0.

§3.1. As indicated in §0, the static vacuum Einstein equations

$$ur = D^2 u, \qquad \Delta u = 0, \tag{3.1}$$

play the fundamental role in the understanding of degenerations of sequences of Yamabe metrics. It is immediate from the definition that these equations are equivalent to the equation

$$L^* u = 0, \tag{3.2}$$

on scalar-flat manifolds. Obviously, there are no non-trivial solutions of the equations (3.1) on closed manifolds, although (3.1) or (3.2) may have locally defined solutions.

We discuss briefly the relation between the equation (3.2) and Einstein metrics on 4-manifolds, since it does not seem to appear in the literature.

Proposition 3.0. *Let g be a metric of constant scalar curvature on a 3-dimensional domain Ω, with smooth boundary $\partial\Omega$. Then any solution of (3.2) on Ω with $u > 0$ on Ω gives an Einstein 4-manifold of the form*

$$X^4 = \Omega \times_u S^1, \tag{3.3}$$

with scalar curvature $s_X = 2s_g$.

Proof. Let $u > 0$ be a solution to (3.2) in Ω. For any form $\eta \in T_g\mathbb{M}$ of compact support in Ω, we then have

$$\int u \cdot L(\eta)\, dV_g = \int \langle L^* u, \eta\rangle dV_g = 0.$$

Now consider the warped product $X^4 = \Omega \times_u S^1$, with metric given by

$$g_X = g + u^2 d\theta^2. \tag{3.4}$$

The volume form of g_X is given by $u dV_g \wedge d\theta$, and one computes, c.f. [Bes, Ch.9J], that the scalar curvature s_X of g_X is given by

$$s_X = s - 2\frac{\Delta u}{u}. \tag{3.5}$$

Thus, applying the divergence theorem

$$\int_X s_X dV_X = \int_\Omega \left(s - 2\frac{\Delta u}{u}\right) u\ dV_g = \int_\Omega u \cdot s\, dV_g - 2\int_{\partial\Omega} \langle du, \nu\rangle dA_g,$$

where ν is the outward unit normal.

Consider a compactly supported 1-parameter variation of g_X, i.e. a curve of metrics $g_X(t)$ on X, with $g_X(0) = g_X$, of the form $g_X(t) = g_t + u_t^2 d\theta^2$. We suppose also that $\mathrm{vol}_{g_X(t)}(X) = \mathrm{vol}_{g_X}(X)$. Then, for $\alpha = \frac{d}{dt} g_X(t)|_{t=0}$, we have at $t = 0$,

$$\frac{d}{dt}\int_X s_X(t)\ dV_t = \frac{d}{dt}\int_\Omega s_t u_t\ dV_t - 2\frac{d}{dt}\int_{\partial\Omega} \langle du_t, \nu\rangle_{g_t}\ dA_{g_t}.$$

Now the boundary term vanishes, since the variation is of compact support. For the first integral, we have

$$\frac{d}{dt}\int_\Omega s_t u_t dV_t = \int_\Omega s' u dV + s\int_\Omega (udV)',$$

where we have used the fact that s is constant. The second integral here vanishes, since the variation is volume preserving, while for the first integral

$$\int_\Omega s' u dV = \int_\Omega L(\alpha) u dV = \int_\Omega \langle L^* u, \alpha\rangle dV = 0. \tag{3.6}$$

Thus the gradient $\nabla\mathcal{S}$ of the unnormalized scalar curvature functional on metrics on X vanishes when paired with all compactly supported, volume preserving, variations for which S^1 acts by isometries. By the so-called symmetric criticality principle, c.f. [Bes, Thm.4.23], it follows that $\nabla\mathcal{S} = 0$ on (X, g_X), i.e. g_X is an Einstein metric. The scalar curvature s_X is determined by (3.5).

Alternately, the equations (1.14) can be used to show that the equation (3.2) is equivalent to the condition that g_X is Einstein.

$\blacksquare$

Of course a similar relation is valid in all dimensions. Conversely, the proof shows that if (X, g_X) is an Einstein 4-manifold of the form (3.4), then (3.2) holds. We note that the Einstein metric on X may (or may not) have singularities if Ω includes points where $u = 0$.

The equations (1.14) for the Ricci curvature of the 4-manifold (X, g_X), together with the basic identity (2.10), show that the Ricci curvature r_X, or the Einstein tensor $r_X - \frac{1}{2}s_X g_X$, of X may be expressed solely in terms of ξ, s and u. In the language of general relativity, (X, g_X) may be viewed as a (Riemannian) space-time with a matter or field term involving only ξ, s and u. In regions where u is bounded away from 0 and ∞, this term is apriori bounded in L^2.

§3.2. Although (3.1) has no non-trivial global solutions on compact manifolds, it is closely related to local degenerations of Yamabe metrics. Thus, suppose $\{x_i\}$ is a sequence in (M, g_i) such that

$$\rho_i(x_i) \to 0. \tag{3.7}$$

Consider the rescaled metrics

$$g_i' = \rho_i(x_i)^{-2} \cdot g_i. \tag{3.8}$$

Throughout §3, we make the following (weak) *non-collapse assumption* on sequences $\{g_i\}$ of Yamabe metrics: for some arbitrary, but fixed $\nu_o > 0$,

$$\nu_i(x) \geq \nu_o \cdot \rho_i(x). \tag{3.9}$$

The inequality (3.9) is scale-invariant, and it implies that in the scale where $\rho(x) \sim 1$, as in (3.8), that one has a uniform lower bound on the volume of geodesic r-balls, for $r \leq 1$, in terms of Euclidean volumes; see §4.1 for some further remarks on the collapse case.

In particular, for the pointed sequence (M, g_i', x_i) above, it follows from Theorem 1.5 that a subsequence of $\{g_i'\}$ converges, modulo diffeomorphisms, in the weak $L^{2,2}$ topology to a limit metric g', defined on a unit ball $B_x'(1)$, centered at $x = \lim x_i$.

The following Proposition proves Theorem A(I).

Proposition 3.1. *Let (M, g_i, x_i) be a pointed sequence of Yamabe metrics on a closed 3-manifold M satisfying (3.7) and (3.9) with $s_{g_i} \geq -s_o$, for some $s_o < \infty$. Then the blow-up limit $(B_x'(1), g', x)$ is an $L^{2,2}$ weak solution of the static vacuum Einstein equations (3.1).*

Proof. Each metric g_i has an associated splitting (2.10), i.e.

$$L^* u + \xi = -\frac{s}{3} \cdot g, \tag{3.10}$$

where we have dropped the subscript i. Such a splitting holds also for the metrics g_i' in (3.8), with the same function $u = u_i$, since u is scale invariant. The scaling properties of curvature imply

$$s_i' = \rho_i(x_i)^2 \cdot s_i \to 0, \tag{3.11}$$

since s_i is uniformly bounded, (c.f. (0.4)), and $\rho_i(x_i) \to 0$, by (3.7). Further, since ξ scales as curvature,

$$\int_M |\xi_i'|^2 dV_i' = \rho_i(x_i) \cdot \int_M |\xi_i|^2 dV_i \to 0, \tag{3.12}$$

since $\int_M |\xi_i|^2 dV_i$ is uniformly bounded, see (2.11). We emphasize that (3.12) uses in an essential way that M is 3-dimensional.

Thus, it follows from (3.10)-(3.12) that

$$(L')^* u_i \to 0, \qquad \text{strongly in } L^2(M, g_i'), \tag{3.13}$$

and hence the limit $L^{2,2}$ metric g' satisfies

$$(L')^* u = 0. \tag{3.14}$$

in $B_x'(1)$. Here, to be precise, we need to examine the limiting behavior of $\{u_i\}$. Suppose first that $\{u_i\}$ is bounded in $L^2(B_i), B_i = B'_{x_i}(1)$. Since then $\Delta u_i \to 0$ in $L^2(M, g_i')$, it follows from standard elliptic estimates, c.f. [GT, Thm.8.8], that $\{u_i\}$ is uniformly bounded in $L^{2,2}$ in $B_i(1-\delta) = B'_{x_i}(1-\delta)$, for any given $\delta > 0$. Hence $\{u_i\}$ sub-converges to a limit function $u \in (L^{2,2})_{loc}$ on $B_x'(1)$ and (3.14) holds weakly, i.e. when paired with smooth 2-tensors of compact support in $B_x'(1)$.

If instead $||u_i||_{L^2(B_i)} \to \infty$ as $i \to \infty$, we just renormalize u_i by setting $\bar{u}_i = u_i/||u_i||_{L^2(B_i)}$. When renormalizing (3.10) by the same factor, all the terms become even smaller and the argument proceeds as before. This renormalization process will recur several times throughout §3.

■

From the discussion in §1.3, weak $L^{2,2}$ solutions of the static vacuum equations are smooth, (at least in regions where the potential function u does not vanish).

The following examples of static vacuum solutions, although trivial, are important in understanding the structure of the arguments to follow.

Examples of Static Vacuum Solutions:

Super-trivial solutions (N, g, u): $u \equiv 0$, (N, g) arbitrary.

Trivial solutions: $(N, g, u) = (\mathbb{R}^3, g_o, u_o)$, where g_o is a flat metric on $\mathbb{R}^3$ and u_o is a constant or affine function. Similarly, one may have such solutions on flat quotients of $\mathbb{R}^3$.

Note that super-trivial solutions give no information whatsoever about the Riemannian manifold (N, g). Thus, in order for Proposition 3.1 to be of any use, one must study the sequence (M, g_i) of Yamabe metrics away from the locus where u_i approaches 0, see also Remark 3.15(i).

We have the following characterization of the trivial or flat solution, generalizing a classical result of Lichnerowicz [Li], (which assumes that u is asymptotically constant).

Theorem 3.2. **(I)** *Let (N, g, u) be a complete solution to the static vacuum equations (3.1), i.e. (N, g) is a complete Riemannian manifold. If $u > 0$ on N, then N is flat, and u is constant.*

(II) *Let (N, g, u) be a solution of (3.1) and let $U \subset N$ be any domain with smooth boundary on which $u > 0$. If $t(x) = \mathrm{dist}_N(x, \partial U)$, for $x \in U$, then there is an absolute constant $K < \infty$ such that*

$$|z|(x) \leq \frac{K}{t^2(x)}, \qquad \text{and} \qquad (u^{-1}|\nabla u|)(x) \leq \frac{K}{t(x)}. \tag{3.16}$$

The constant K does not depend on the domain U, (provided $u > 0$ on U), or on the static vacuum solution (N, g).

Proof. The proof is deferred to the Appendix, since the methods do not bear directly on the main discussion to follow. (The proof of (3.16) is similar though to the proof of Theorem 3.3 below).

The relation between the non-existence (of non-trivial solutions) in (I) is closely related to the existence of the pointwise curvature estimate (3.16) in (II). This situation occurs frequently in

geometric P.D.E.'s and statements of the form (I) and (II) are often equivalent. Thus, given (I), one obtains (II) by a basically standard scaling argument. Conversely, (II) immediately implies (I) since the function $t \equiv \infty$ in this case.

■

As noted before, the canonical solution of the static vacuum equations (3.1) is the Schwarzschild metric (0.17). This metric is characterized by the conditions in Theorem 1.1.

§3.3. In this subsection, we show that the curvature is controlled in L^2 in regions of (M, g) where the level sets of $u = 1 + f$ do not come too close together. Let

$$L^c = \{x \in M : u(x) = c\} \quad \text{and} \quad U^c = \{x \in M : u(x) > c\},$$

denote the c-level and super-level sets of u.

Theorem 3.3. *Let g be a Yamabe metric on a closed 3-manifold M, of volume 1, satisfying $s_g \geq -s_o > -\infty$ and (3.9), i.e. $\nu(x) \geq \nu_o \cdot \rho(x)$. Given any constant $c > 0$, there is a constant $\rho_o = \rho_o(c, s_o, \nu_o) > 0$, such that, for any $x \in U^c$,*

$$\rho(x) \geq \rho_o \cdot \min\{1, \operatorname{dist}(x, L^c)\}. \tag{3.17}$$

Proof. Note that the estimate (3.17) is exactly the L^2 analogue of the estimate (3.16), but pertains to quite general Yamabe metrics while (3.16) holds only for the much more rigid class of static vacuum solutions. The proof of (3.17) reduces to that of (3.16) or Theorem 3.1(I) by taking blow-up limits.

Thus, we assume (3.17) is false, and will derive a contradiction. Given $c > 0$, $s_o < \infty$, if (3.17) does not hold, then there is a sequence of Yamabe metrics γ_k on M_k such that $\mathrm{vol}_{\gamma_k} M_k = 1$, $s_{\gamma_k} \geq -s_o$, and

$$\frac{\rho_k(x_k)}{\min\{v^{1/3}, \operatorname{dist}_{\gamma_k}(x_k, L^c)\}} \to 0, \qquad \text{as } k \to \infty, \tag{3.18}$$

for some sequence of points $x_k \in U^c = U^c(k)$; here, we replace the constant 1 in (3.17) by $v^{1/3} = \mathrm{vol}_{\gamma_k} M_k^{1/3} = 1$, so that the equation (3.18) is scale invariant. Choose points y_k realizing the minimum of the ratio in (3.18); it follows that $\rho_k(y_k) \to 0$, as $k \to \infty$. We rescale the metrics to make $\rho_k(y_k)$ of size 1. Thus, define metrics $\gamma_k' = \rho_k(y_k)^{-2}\gamma_k$ and consider the sequence of pointed Riemannian manifolds (U^c, γ_k', y_k). By construction

$$\rho_k'(y_k) = 1, \tag{3.19}$$

and

$$\mathrm{vol}_{\gamma_k'} M_k \to \infty, \qquad \operatorname{dist}_{\gamma_k'}(y_k, L^c) \to \infty, \qquad \text{as } k \to \infty. \tag{3.20}$$

Also, for any sequence $z_k \in U^c$, the estimate

$$\rho_k'(z_k) \geq \rho_k'(y_k) \frac{\operatorname{dist}_{\gamma_k'}(z_k, L^c)}{\operatorname{dist}_{\gamma_k'}(y_k, L^c)}, \tag{3.21}$$

follows from the fact that the ratio (3.18) is scale invariant and is minimized at y_k. Thus, the sequence (U^c, γ_k', y_k) has L^2 curvature radius uniformly bounded below, at points within a bounded but arbitrary distance from y_k.

Next, recall that the volume radius (1.15) scales as a distance. By (3.9), (3.21) and scale invariance, we have a uniform lower bound $\nu'_k(z_k) \geq \nu_o \cdot \rho'(z_k) \geq \nu_o \cdot C(\mathrm{dist}(z_k, y_k))$. Thus the sequence (U^c, γ'_k, y_k) cannot collapse anywhere. From Theorem 1.5, it follows that this pointed sequence has a subsequence which converges uniformly on compact subsets, in the *weak* $L^{2,2}$ topology, to a limit $L^{2,2}$ Riemannian manifold (N, γ', y). The estimate (3.20) further implies that the metric γ' is complete; the distance to the boundary of U^c, namely L^c, goes to infinity as $k \to \infty$.

From the arguments in Proposition 3.1, c.f. (3.10)-(3.13), we see that

$$(L')^* u_k \to 0, \qquad \text{strongly in} L^2(\gamma'_k). \tag{3.22}$$

Now by definition, $u_k \geq c$ in U^c. If $u_k(y_k) \to \infty$, consider the function

$$\bar{u}_k(z_k) = \frac{u_k(z_k)}{u_k(y_k)}, \tag{3.23}$$

so that $\bar{u}_k \geq 0$, $\bar{u}_k(y_k) = 1$. Otherwise, let $\bar{u}_k = u_k$. The trace equation associated to (3.22) reads

$$\Delta' \bar{u}_k + s'_k \bar{u}_k = (u_k(y_k))^{-1}(\mathrm{tr}\ \xi'_k + s'_k). \tag{3.24}$$

Note that the right-hand side of (3.24) goes to 0 in L^2. It follows from the lower bound $\bar{u}_k \geq 0$ and the DeGiorgi-Nash-Moser estimates, c.f. [GT, Thms 8.17, 8.18, 8.22], that the oscillation of $\bar{u}_k$ is uniformly bounded in $B'_{y_k}(\frac{1}{2})$. Applying the same considerations to neighboring balls of radius $\frac{1}{2}\rho'$, it follows that the oscillation of $\bar{u}_k$ is uniformly bounded in $B'_{z_k}(\frac{1}{2}\rho'(z_k))$ for all z_k within uniformly bounded distance to y_k. This, together with standard L^2 estimates for elliptic equations, c.f. [GT, Thm.8.8] imply that a subsequence of $\{\bar{u}_k\}$ converges, in the weak $L^{2,2}$ topology, with respect to the metrics γ'_i, to a limit $L^{2,2}$ function $\bar{u}$ on N. Further, by (3.22) and the fact that (3.22) is preserved under the renormalization (3.23), the limit pair $(\gamma', \bar{u})$ satisfies

$$L'^* \bar{u} = 0, \qquad \Delta' \bar{u} = 0, \tag{3.25}$$

so that $(N, \gamma', \bar{u})$ is a weak solution of the static vacuum Einstein equations.

Note that the (weakly) harmonic function $\bar{u}$ satisfies $\bar{u} \geq 0$ everywhere and $\bar{u}(y) = 1$. By the (weak) maximum principle, c.f. [GT, Thm.8.1], in fact $\bar{u} > 0$ everywhere. As noted in §1.3, when the potential $\bar{u} > 0$, elliptic regularity implies that $L^{2,2}$ weak solutions of (3.25) are C^∞. From Theorem 3.2(I), it follows that γ' is flat, and $\bar{u}$ is constant.

On the other hand, we have the estimate (3.19) for the sequence (γ'_k, y_k). The fact that the convergence of γ'_k to γ' is only in the weak $L^{2,2}$ topology does not imply that (3.19) passes continuously to the limit. However, we will show in Theorem 3.4 below that in fact the convergence $\gamma'_k \to \gamma'$ is in the strong $L^{2,2}$ topology. By (1.28), the radius $\rho(x)$ is continuous in the strong $L^{2,2}$ topology and one thus obtains the estimate

$$\rho'(y) = 1,$$

on the limit. This however contradicts the fact that γ' is flat; a complete flat manifold (N, g) clearly has $\rho(x) = \infty$, at any x.

Hence the proof of Theorem 3.3 is completed with the proof of the following result, which will also be used frequently in the work to follow.

■

Theorem 3.4 (Strong Convergence). *Let $\{g_i\}$ be a sequence of Yamabe metrics, (not necessarily of volume 1), with associated splittings*

$$L^*u_i + \xi_i = -\frac{s_i}{3} \cdot g_i. \tag{3.26}$$

Suppose that for all i, there is a constant $d > 0$ such that

$$r_h(x_i, g_i) \geq 1, \quad \text{and} \quad r_h(y_i, g_i) \geq d, \quad \forall y_i \in \partial B_{x_i}(1), \tag{3.27}$$

where r_h denotes the $L^{2,2}$ harmonic radius. Suppose also that $\xi_i \to 0$ strongly in $L^2(B_{x_i}(1)), s_i \geq -s_o$, for some s_o, and that there is a constant $u_o > 0$ such that

$$u_i \geq u_o \ \text{ on } B_{x_i}((1 + \tfrac{1}{2}d)). \tag{3.28}$$

Then on $B_{x_i}(1)$, a subsequence of $\{g_i\}$ converges strongly in the $L^{2,2}$ topology to a limit metric g_o on $B_x(1)$, where $x = \lim x_i$.

Proof. Let $B = B_i = B_{x_i}(1)$ and set $B' = B_{x_i}(1 + \frac{d}{2})$. By (3.27) and Theorem 1.5, we may assume that a subsequence of $\{g_i\}$ converges weakly in the $L^{2,2}$ topology to a limit metric g_o on B'. Thus, there are coordinates $\{y_k\}$ on suitable balls $D \subset B_{x_i}(1 + d)$, covering B', such that the functions

$$(g_i)_{kl} \to (g_o)_{kl}, \tag{3.29}$$

weakly in $L^{2,2}(D)$. As in the proof of Theorem 3.3, (by dividing by $u(x_i)$ if necessary, see the discussion following (3.23)), we may assume that the functions u_i are uniformly bounded, in $L^\infty(B')$, and that (3.28) holds, possibly with a different constant u_o. In particular, a subsequence converges strongly in L^2 to a limit function u. Thus, from the trace equation

$$\Delta_i u_i = \text{tr } \xi_i - s_i u_i + s_i, \tag{3.30}$$

and the arguments above, together with the assumption on $\xi, \Delta_i u_i$ converges strongly in L^2 to a limit L^2 function $\Delta_o u$ on B' with $u \in T^{2,2}(B')$.

We now use the L^2 estimates for elliptic equations of the form (3.30), see [GT, Thm 8.8]. This gives

$$||D^2 u_i||_{L^2(B)} \leq C(||\Delta_i u_i||_{L^2(B')} + ||u_i||_{L^2(B')}), \tag{3.31}$$

where C depends on the $L^{2,2}$ norm of the metrics g_i and the constant d in (3.27). Here the norms and derivatives D^2 are taken with respect to the fixed coordinates $\{y_k\}$ and limit metric g_o. From this, it then follows first that

$$||D^2 u_i||_{L^2(B)} \leq C, \tag{3.32}$$

so that a subsequence of $\{u_i\}$ converges weakly in the $L^{2,2}$ topology to u. By the Sobolev embedding theorem, $\{u_i\}$ (sub)-converges strongly in the $L^{1,2}$ topology. Repeating the estimate (3.31) on $u - u_i$ gives

$$||D^2(u - u_i)||_{L^2(B)} \leq C(||\Delta_i(u - u_i)||_{L^2(B')} + ||(u - u_i)||_{L^2(B')}). \tag{3.33}$$

Clearly $||(u - u_i)||_{L^2(B')} \to 0$. Write $\Delta_i(u - u_i) = (\Delta_i u - \Delta_o u) + (\Delta_o u - \Delta_i u_i)$. From preceding arguments, $\Delta_i u_i \to \Delta_o u$ strongly in $L^2(B')$. Consider then the sequence $\Delta_i u - \Delta_o u$. We have in the y_k-coordinates

$$\Delta_i u = g_i^{kl} \partial_{kl} u + \partial g_i^{kl} \partial_k u. \tag{3.34}$$

Since $\{g_i\}$ is uniformly bounded in $L^{2,2}$, a subsequence converges strongly in $L^{1,2} \cap L^{\infty}$ to g_o. This shows that $\Delta_i u$ converges strongly in L^2 to $\Delta_o u$. Thus, (3.33) implies that

$$||D^2(u - u_i)||_{L^2(B)} \to 0. \tag{3.35}$$

If D_i^2 denotes the Hessian with respect to the metrics g_i, the same reasoning as above on the Laplacian in (3.34) then also gives

$$D_i^2 u_i \to D^2 u, \qquad \text{strongly in } L^2(B). \tag{3.36}$$

Now return to the splitting equation (3.26), which we write as

$$u_i r_i = D_i^2 u_i - \Delta_i u_i + \xi_i + \frac{s_i}{3} \cdot g_i. \tag{3.37}$$

By the preceding arguments, the right hand side of (3.37) converges strongly in L^2 to its limit on B and thus

$$\int_B |u \cdot r_o - u_i \cdot r_i|^2 dV_o \to 0, \tag{3.38}$$

where the norm is taken in the g_o metric and r_o is the weak L^2 limit of the Ricci curvature r_i of g_i. Since $u_i \to u$ in $L^{2,2} \cap C^o$ on B and since $u_i \geq u_o$ by (3.28), it follows that

$$\int_B |r_o - r_i|^2 dV_o \to 0, \tag{3.39}$$

so that the Ricci curvature of $\{g_i\}$ converges strongly in L^2 to the Ricci curvature of g_o on B. The same reasoning shows that r_i converges strongly to r on a thickening, say $B(1 + \frac{d}{4})$ of B. This implies the strong $L^{2,2}$ convergence of g_i to g_o, via the equation for the Ricci curvature in harmonic coordinates on B, see [An3, p.434] for the details here.

■

Remark 3.5. (i): If one drops the assumption (3.27) on the behavior of r_h at the boundary, so that the sequence $(B_{x_i}(1), g_i)$ satisfies $r_h(x_i, g_i) = 1$ with $u_i \geq u_o$ on $B_{x_i}(1)$ and $\xi_i \to 0$, $s_i \geq -s_o$, then Theorem 3.4 implies that a subsequence converges strongly in the $L^{2,2}$ topology on $B_{x_i}(1 - \delta)$, for any fixed $\delta > 0$, to a limit metric g_o on $B_x(1 - \delta)$.

(ii): Returning to Theorem 3.3, note that the estimate (3.17), together with Lemma 1.4, implies that if $u_i \geq u_o > 0$ on M, then

$$\int_M |z_i|^2 dV_i \leq C/u_o. \tag{3.40}$$

This has already been proved by elementary means in (2.18).

(iii): We note that exactly the same proof as Theorem 3.3 shows the estimate (3.17) holds in the region $U_{-c} = \{x \in M : u < -c < 0\}$, i.e. $\rho(x) \geq a \cdot \mathrm{dist}(x, L^{-c})$, for $x \in U_{-c}$.

§3.4. The discussion in §3.3 implies that $\rho(x_i)$ can become arbitrarily small, under the sequence $\{g_i\}$, only if the 0-level set L^o, or an ε-level set L^{ε} for ε very small, comes arbitrarily close to x_i.

In this subsection, we will begin the study of the geometry of the degeneration of (M, g_i) in a neighborhood of the 0-level L^o. On the other hand, from the remarks above, the 0-level itself must be avoided; see also Remark 3.15(i).

The treatment of the three possible cases $\sigma(M) < 0$, $\sigma(M) = 0$ and $\sigma(M) > 0$ is somewhat different. Some indication for the need to treat these cases separately is already apparent from §2. Thus, throughout §3.4, we assume

$$\sigma(M) < 0,$$

or more precisely that $\{g_i\}$ is a sequence of unit volume Yamabe metrics on M with scalar curvature bounded away from 0 and $-\infty$, i.e.

$$-\infty < -s_o \leq s_{g_i} \leq -s_1 < 0. \tag{3.41}$$

The cases $\sigma(M) = 0$ and $\sigma(M) > 0$ are treated in §3.5 and §3.6 respectively. As in Theorem A(II), c.f. (0.19), we also assume

$$\int_M |z^T|^2 dV \leq K, \tag{3.42}$$

and

$$\nu(x) \geq \nu_o \cdot \rho(x); \tag{3.43}$$

note that (3.43) is more general than (0.18).

To set the stage for the considerations to follow, let $T_i = \max u_i$ and choose x_i such that

$$|u_i(x_i)/T_i - 1| \to 0, \qquad \text{as } i \to \infty. \tag{3.44}$$

Thus, we are considering points as far away as possible, in terms of the function u, from the 0-levels of u. By (2.35), as discussed in Remark 2.12, (3.41) and (3.42) imply one has a uniform lower bound

$$T_i \geq \lambda_o \geq 0. \tag{3.45}$$

Note: The bound (3.45) is the only place in §3.4, (and also in §3.6), where the assumption (3.42) is used. In §3.5, corresponding to the case $\sigma(M) = 0$, (3.42) is used in a stronger way.

Consider $\rho(x_i)$, the L^2 curvature radius at x_i, of course with respect to the metric g_i. If there exists a uniform lower bound

$$\rho(x_i) \geq \rho_o, \quad \forall i, \tag{3.46}$$

then Theorem 1.5 provides a complete description of the possible behavior of $\{g_i\}$ in $B_{x_i}(\rho_o)$. Suppose instead, for a possibly different choice $\{x_i^1\}$, that both

$$\begin{gathered} |u_i(x_i^1)/T_i - 1| \to 0, \text{ and} \\ \rho(x_i^1) \to 0, \quad \text{as } i \to \infty. \end{gathered} \tag{3.47}$$

Thus, the metrics g_i are degenerating in a (progressively smaller) neighborhood of x_i^1 and this degeneration cannot be described by Theorem 1.5.

To understand the degeneration of (M, g_i) at or near $\{x_i^1\}$, rescale the metrics g_i by $\rho(x_i^1)^{-2}$. Thus, set

$$g_i^1 = \rho(x_i^1)^{-2} \cdot g_i, \tag{3.48}$$

and let ρ^1 denote the L^2 curvature radius with respect to the rescaled metrics g_i^1. One then has

$$\rho^1(x_i^1) = 1, \qquad \text{for all } i, \tag{3.49}$$

so that Theorem 1.5 and (3.43) imply the sub-convergence of $\{g_i^1\}$ in $B_{x_i^1}^1(1)$. Thus we may assume that $\{g_i^1\}$ converges, weakly in the $L^{2,2}$ topology, to a limit $L^{2,2}$ metric g^1, defined on $B^1 = B_{x^1}^1(1)$, with base point $x^1 = \lim x_i^1$.

Proposition 3.6. *The limit (B^1, g^1, x^1) is a flat solution of the static vacuum equations (3.1), with potential function*

$$\bar{u} = \lim(u_i/T_i) \equiv 1 \quad \text{on } B^1. \tag{3.50}$$

Proof. The splitting (2.10) of g_i, when rescaled, gives the splitting for g_i^1. Thus

$$\begin{aligned} (L^*)^1(u_i) + \xi_i^1 &= -\tfrac{s_i^1}{3} \cdot g_i^1, \\ \Delta^1 u_i + s_i^1 u_i &= \mathrm{tr}\ \xi_i^1 + s_i^1. \end{aligned} \tag{3.51}$$

As in the proof of Proposition 3.1, we have $\xi_i^1 \to 0$ in $L^2(M, g_i^1)$, $s_i^1 \to 0$ in $L^\infty(M, g_i^1)$. While it may be possible apriori that $u_i(x_i^1) \to +\infty$, (corresponding to the possibility that $T_i \to \infty$), as in the proof of Theorem 3.3, (or Proposition 3.1) we renormalize equation (3.51) by dividing T_i to obtain the (renormalized) limit function $\bar{u}$ on B^1. As noted above, T_i is bounded away from 0 by (3.45), (a consequence of the standing assumption (3.42)), so that the renormalizations of the terms ξ_i^1, s_i^1 go to 0 at least as fast as before.

Thus, on B^1, the limit metric g^1 satisfies the static vacuum Einstein equations

$$\begin{aligned} (L^*)^1(\bar{u}) &= 0, \\ \Delta^1 \bar{u} &= 0, \end{aligned} \tag{3.52}$$

where $\bar{u} = \lim u_i/T_i$. As in the proof of Theorem 3.3, the convergence of $\bar{u}_i = u_i/T_i$ to the limit is in the weak $L^{2,2}$ topology, and since $\bar{u}_i(x_i^1) = 1$, the limit $\bar{u}$ is a weakly harmonic $L^{2,2}$ function, which is not identically 0.

As discussed in §1.3, the equations (3.52) imply in particular that the metric g^1 is smooth away from the 0-locus of $\bar{u}$. Now by (3.44), x^1, the center of the ball B^1, satisfies

$$1 = \bar{u}(x^1) \geq \bar{u}(y), \tag{3.53}$$

for all $y \in B^1$. Since $\bar{u}$ is harmonic, it follows that $\bar{u} \equiv 1$ and thus (B^1, g^1) is flat.

∎

The convergence $g_i^1 \to g^1$ is in the strong $L^{2,2}$ topology, on $B_{x_i^1}^1(s)$, for any fixed $s < 1$, by Theorem 3.4. It thus follows that all of the curvature of $\{g_i^1\}$ in L^2 is concentrating on the boundary $\partial B_{x_i^1}^1(1)$. This implies that the limit (B^1, g^1) itself does not yet effectively model the degeneration of g_i near x_i^1; the base points x_i^1 must be altered slightly.

The fact that some or all of the curvature in L^2 is concentrating on $\partial B_{x_i^1}^1$ implies, (and is equivalent to),

$$\rho^1(y_i) \to 0,$$

for some sequence $y_i \in \partial B^1_{x_i^1}(1)$, or, what is same, $\rho_i(y_i) << \rho_i(x_i^1)$. Thus the curvature is blowing up at y_i much faster than at x_i^1. This is the first indication that the curvature of (M, g_i) blows up at many different scales.

The preceding remarks lead naturally to the following definitions.

Definition 3.7. A point y in a complete Riemannian manifold (N, γ) is (ρ, c) *buffered* if $c > 0$ and

$$\frac{\rho(y)^4}{\mathrm{vol}\, B_y(\rho(y))} \int_{B_y((1-c)\cdot\rho(y))} |r|^2 \geq c \cdot c_o, \tag{3.54}$$

where c_o is the constant in the definition of ρ, c.f. (1.26).

Similarly, y is *strongly* (ρ, d) *buffered* if $d > 0$ and

$$\rho(z) \geq d \cdot \rho(y), \tag{3.55}$$

for all $z \in \partial B_y(\rho(y))$.

A sequence of points y_i in the manifolds (N, γ_i) is (ρ, c) buffered, or strongly (ρ, c) buffered, if each y_i is.

The buffer constants c and d are arbitrary small parameters, as is the parameter c_o; their precise values, beyond being small, are not important. From §1, c_o is a fixed small number throughout the paper, e.g. $c_o = 10^{-3}$. The buffer constant c, (more important than d), might be allowed to vary over small numbers, say $0 < c < 10^{-3}$, but in any given discussion, the value of c will be fixed.

Note that the strong (ρ, d) buffered condition appears in Theorem 3.4 (strong convergence), and is invariant under scaling. Thus, if $\{g_i\}$ is a sequence of unit volume Yamabe metrics on a closed 3-manifold M with $\rho(y_i) \to 0$, then the blow up metrics $g_i' = \rho(y_i)^{-2} \cdot g_i$ converge strongly in $L^{2,2}$ on $B'_{y_i}(1)$, to a non-flat limit, provided the sequence $\{y_i\}$ is strongly (ρ, d) buffered, and u_i is bounded away from 0 on $B'_{y_i}(1)$. Thus, this condition prevents *any* of the curvature from concentrating in L^2 (with normalized measure) on the boundary.

Similarly, the (ρ, c) buffered condition (3.54) is invariant under scaling and prevents *all* of the curvature from concentrating in L^2 on the boundary. Thus, if c is close to 0, then a definite (small) percentage of the curvature in L^2 is in the ball of radius $(1-c)$. (The situation where c is close to 1 will never arise here). In particular, from Theorem 3.4, any limit $B'_y(1)$ of a (ρ, c) buffered sequence $(B'_{y_i}(1), g_i', y_i)$ of Yamabe metrics as above, with u_i bounded away from 0 in $B'_{y_i}(1)$, cannot be flat, since one has strong convergence everywhere in the interior. Conversely, again by Theorem 3.4, if u_i is bounded away from 0 on $B'_{y_i}(1)$ and the limit is not flat, then the sequence is (ρ, c) buffered, for some $c > 0$, c.f. also Remark 3.5(i).

Note that from the definition, it is obvious that if a given sequence is (ρ, c) buffered, then it is (ρ, c') buffered, for any $c' < c$, and similarly for strongly buffered sequences.

The next Lemma formalizes the relation between these notions, for Yamabe metrics.

Lemma 3.8. *Let g be a unit volume Yamabe metric on a closed 3-manifold M, and y a base point in M which is strongly (ρ, d) buffered. Suppose $u(y) = 1$, the non-collapse assumption (3.43) holds at y, and the blow-up $(B'_y(1), g', y)$, $g' = \rho(y)^{-2} \cdot g$ is ϵ-close to a static vacuum solution, in the sense that $||L^*u||_{L^2} \leq \epsilon$ in $B'_y(1)$, as in (3.13).*

Under these assumptions, there exists $\varepsilon_o > 0$ such that if $\varepsilon \leq \varepsilon_o$, and if either

(i) u_i is bounded away from 0 in $B'_{yi}(1)$, or

(ii) $|u_i - 1| \leq \frac{1}{4}$ *in* $B'_{yi}(\frac{1}{2})$,
then there is a constant $c = c(d, c_o, \nu_o) > 0$ *such that* y *is* (ρ, c) *buffered.*

Proof. The proof is by contradiction. Thus, if the conclusion is not true, there exists a pointed sequence (M, g_i, y_i) satisfying the assumptions with $\varepsilon = \varepsilon_i \to 0$, such that the sequence y_i is not (ρ, c) buffered. Let $(B'(1), g', y)$ be a weak $L^{2,2}$ limit of (a subsequence of) $(B'_{y_i}(1), g'_i, y_i)$, so that $(B'(1), g')$ is an $L^{2,2}$ static vacuum solution. The existence of such limits follows, as previously in §3.4, from the non-collapse assumption (3.43).

By the discussion above regarding strong convergence, it suffices to prove that u_i is bounded away from 0 on $B'_{y_i}(1)$ and the limit is not flat. In the case of assumption (i), this first condition is obvious, while the second follows since ρ is continuous under strong $L^{2,2}$ convergence. Hence one has a contradiction in this case.

With regard to (ii), suppose first the limit $(B'(1), g')$ were flat. Then from the static vacuum equations, the limit function u is an affine function on $B'(1)$. The assumption (ii) then implies that u is uniformly bounded away from 0 in $B'(1)$, in fact $u \geq \frac{1}{2}$. Further, since y_i is strongly (ρ, d) buffered, the functions u_i are uniformly controlled in $L^{2,2}$, and hence controlled in C^{α}, $\alpha < \frac{1}{2}$, in $B'_{y_i}(1 + \frac{d}{2})$. It follows that u_i is uniformly bounded away from 0 in $B'_{y_i}(1)$. As in Case (i) above, this implies that the full limit $B'(1)$ could not have been flat.

Since the full limit $(B'(1), g')$ is not flat, it follows from the real-analyticity of static vacuum solutions discussed in §1.3, and (ii) again, that the limit is not flat in any domain in $B'(1)$ on which u is bounded away from 0; in particular, this is the case on $B'(\frac{1}{2})$. Hence, as above, u_i is bounded away from 0 in $B'_{y_i}(\frac{1}{2})$, and the strong convergence to this limit $B'(\frac{1}{2})$ implies that the sequence is (ρ, c) buffered, for some $c > 0$.

■

The buffer constant $c = c(d, c_o, \nu_o)$ in Lemma 3.8 could be explicitly estimated in terms of d, c_o and ν_o; however, we will have no need to do this. Of course, c may well be much smaller than d. In the following, we will usually use the contrapositive of Lemma 3.8, namely that if a given sequence y_i is not (ρ, c) buffered, for a given $c > 0$, then it is not strongly (ρ, c) buffered if either (i) or (ii) in Lemma 3.8 hold; here $d = d(c, c_o, \nu_o)$ may be large compared with c, but may made arbitrarily small by choosing c sufficiently small.

Now we return to the situation preceding Proposition 3.6. Of course the points x_i^1 in (3.47) above are not (ρ, c) buffered, for any choice of $c > 0$. Observe that all the positive level sets L^s, $s > 0$, of $\bar{u}_i$ have points converging to $\partial B_{x_i^1}^1(1)$. For otherwise, by Theorem 3.3, $\{x_i^1\}$ would be a (strongly) buffered sequence, a contradiction.

Let $U = U(i)$ be the component of $(u_i/T_i)^{-1}((0, 2]) \subset M$ containing x_i^1. Let $L^k = L^k(i)$, $k \geq 1$, denote the level set $(u_i/T_i)^{-1}(2^{-k+1})$ in $U(i)$, (possibly having many components). By a slight perturbation of the values, we may assume that L^k are smooth hypersurfaces. Let $U^k = U^k(i)$ be the set $u_i^{-1}[2^{-k+1}, 2] \subset U$. The choice of the factor 2 here is for convenience, and may be replaced by any fixed number > 1.

In Theorem 3.10 below, we will construct a (ρ, c)-buffered sequence from an initial sequence $\{x_i^1\}$ satisfying conditions similar to (3.47) above, in order to produce a non-flat blow up limit. This is done essentially by 'descending down' the level sets L^k of u_i, for i fixed.

To understand the underlying idea and motivation of the proof of Theorem 3.10, it is useful to point out that the descent down the levels of u must require an infinite process. This is shown by the following remark and example. Remark 3.9 and Example 3.9 below are not logically necessary in this respect, and may be skipped if preferred.

Remark 3.9. Recall again that the metrics g_i^1 converge uniformly on compact subsets of $B^1_{x_i^1}(1)$ to the flat metric, and $u_i/T_i \to 1$ uniformly on such compact subsets. It is clear that there exists a sequence $\{x_i^2\} \in U^2 \cap B^1_{x_i^1}(1)$ such that

$$\rho^1(x_i^2) \to 0, \quad \text{i.e. } \rho(x_i^2) << \rho(x_i^1), \tag{3.56}$$

and consequently

$$\mathrm{dist}_{g_i^1}(x_i^2, \partial B^1_{x_i}(1)) \to 0.$$

There are many possible choices of $\{x_i^2\}$ satisfying (3.56).

For reasons that will be apparent below, we require further that $\{x_i^2\}$ satisfy the following: $r_i^2 \equiv \mathrm{dist}_{g_i^1}(x_i^2, x_i^1) \to 1$ as $i \to \infty$, and

$$|u_i(x) - u_i(x_i^1)| \leq \mu_o \cdot u_i(x_i^1), \quad \forall x \in B^1_{x_i^1}(r_i^2). \tag{3.57}$$

Here μ_o is a fixed constant, $0 < \mu_o < \frac{1}{2}$, for example $\mu_o = \frac{1}{4}$. Again it is clear that there are many possible choices of such points x_i^2. If $T_i = \max u_i \to \infty$, (3.57) is assumed to apply to the renormalized functions u_i/T_i, as before. The factor T_i will be ignored below.

Now rescale the metrics g_i^1 further to make $\rho(x_i^2) = 1$, i.e. set

$$g_i^2 = \rho^1(x_i^2)^{-2} \cdot g_i^1 = \rho(x_i^2)^{-2} \cdot g_i.$$

Consider then the sequence of pointed Riemannian manifolds $(B^2_{x_i^2}(1), g_i^2, x_i^2)$. This has L^2 curvature radius equal to 1 at x_i^2, and as above, one may apply Theorem 1.5. The collapse case is ruled out by assumption (3.43).

Thus a subsequence of $\{g_i^2\}$ converges, in the weak $L^{2,2}$ topology on $B^2_{x_i^2}(1)$, and in the strong $L^{2,2}$ topology on $B^2_{x_i^2}(s)$, any $s < 1$ away from the locus $\{|u_i| \leq \varepsilon\}$ for any given $\varepsilon > 0$, to a limit (B^2, g^2, x^2). Necessarily, this limit is a solution of the static vacuum Einstein equations (3.1), with limit function u^2. Note that since $u_i(x_i^2) \geq 1 - 2\mu_o$ for i large, the limit function u^2 is not identically 0.

However, we claim that the limit (B^2, g^2) is also flat and u^2 is constant. To see this, consider first the 'half-ball' $B^2_{x_i^2}(1) \cap B^2_{x_i^1}(R_i^2)$, where $R_i^2 = \mathrm{dist}_{g_i^2}(x_i^2, x_i^1)$. Note that in this scale, $R_i^2 \to \infty$ as $i \to \infty$ by (3.56)-(3.57). Since the curvature of $B^1_{x_i^1}(1)$ is bounded, and g_i^2 is a 'blow-up' of g_i^1, the curvature of (D_i^2, g_i^2) goes to 0 uniformly in L^2. Hence the limit half-ball (D^2, g^2) is flat.

To extend this to the full ball B^2, the static vacuum equations imply that $D^2u^2 = 0$, so that u^2 must be an affine function on its domain. We claim that the limit harmonic function u^2 on D^2 is constant, i.e.

$$u^2 = const. \geq 1 - \mu_o. \tag{3.58}$$

To see this, return to the balls $B^2_{x_i^2}(1)$ in the scale g_i^2. In this scale, the previous ball $(B^1_{x_i^1}(1), g_i^1)$ is expanded into a ball $B^2_{x_i^1}(R_i)$ of radius $R_i \to \infty$ as $i \to \infty$. Let γ_i^2 be a minimizing geodesic from x_i^2 to x_i^1 in $B^2_{x_i^1}(R_i)$, parametrized by arclength w.r.t. g_i^2. Note that the length of γ_i^2 becomes

arbitrarily large. Now suppose that u_i^2 converges to a non-constant affine function u^2 on the limit B^2. Note that all the (horo)-balls $B^2_{\gamma_i(t)}(t)$, centered at $\gamma_i(t)$ and containing x_i^2 at the boundary, also converge to flat balls in the limit, for any given t. (Here we recall from the definition of ρ, c.f.(1.27), that $\rho^2(x) \geq \mathrm{dist}_{g_i^2}(x, \partial B^2_{x_i^1}(R_i))$. It follows that the functions u_i approximate the unique non-constant affine function u^2 in $B^2_{\gamma(t)}(t)$ extending the limit function u^2 in B^2. From the assumption (3.57) on the choice of x_i^2, it follows that $|u_i - 1| \leq 2\mu_o$ everywhere in $B^2_{\gamma_i(t)}(t)$ and hence the same holds on the limit. But clearly a non-constant affine function cannot have this property. This proves second claim above.

Now (B^2, g^2, u^2) is an $L^{2,2}$ static vacuum solution, and from the above, (D^2, g^2) is flat, with u^2 a positive constant. Since, from the discussion in §1.3, $L^{2,2}$ static vacuum solutions are real-analytic away from the 0-locus of the potential, it follows that (B^2, g^2) itself is flat, and u^2 is constant in B^2, as claimed.

Thus the geometry of the sequence $(B^2_{x_i^2}(1), g_i^2, x_i^2)$ and that of the potential function u_i on $B^2_{x_i^2}(1)$ is exactly the same as on the previous sequence $(B^1_{x_i^1}(1), g_i^1, x_i^1)$. Clearly one may repeat this process an arbitrary finite number of times; just return to (3.56)-(3.58) and raise the indices successively by 1. However, each limit g^k constructed in this way is flat with constant potential function and hence the sequence $\{x_i^k\}$ is not (ρ, c) buffered, for any fixed $c > 0$. Thus, no finite iteration of this process leads to a (ρ, c) buffered sequence.

It is worthwhile to examine the construction in Remark 3.9 on a specific example.

Example 3.9 Let (N, g) be the Schwarzschild metric (0.17), the canonical solution of the static vacuum equations. Let x_j^1 be a divergent sequence of points in N, so that $t(x_j^1) \to \infty$, where t is the distance to the event horizon. Note that the curvature z of (N, g) decays on the order of t^{-3} as $t \to \infty$ and $u(x_j^1) \to 1 = \max u$.

Now blow-down the metric g based at $\{x_j^1\}$, i.e. form the 'tangent cone at infinity' based at $\{x_j^1\}$, by considering the metrics $g_j^1 = \rho(x_j^1)^{-2} \cdot g$. The metrics g_j^1 converge to the flat metric on $B^1_{x^1}(1), x^1 = \lim x_j^1$ and the functions u_j^1 given by the restriction of u to $B^1_{x_j^1}(1)$ converge to the constant function 1.

Of course the sequence $(B^1_{x_j^1}(1), g_j^1, x_j^1)$ is not (ρ, c) buffered, for any $c > 0$. A short computation, using the cubic curvature decay and the definition of ρ, shows that

$$\rho(x_j^1) \sim t(x_j^1) - (t(x_j^1))^{1/3}.$$

Thus, as in the construction in Remark 3.9, consider a second sequence of points x_j^2, with $t(x_j^2) \sim (t(x_j^1))^{1/3}$. Again if one blows-down the metric g based at $\{x_j^2\}$, then the metrics $g_j^2 = \rho(x_j^2)^{-2} \cdot g$ converge to the flat metric and the functions u_j^2, now given by the restriction of u to $B^2_{x_j^2}(1)$ again converge to the constant function 1.

This process may be repeated any finite number of times, so that by induction, points x_j^k are chosen satisfying

$$t(x_j^k) \sim (t(x_j^1))^{1/3^k}.$$

However one always obtains flat limits in this manner. Observe that for any given choice of k, the functions u_j^k defined as above, always converge to the constant function 1. In fact $u(x_j^k) \sim 1 - t^{-1}(x_j^k) \to 1$ as $j \to \infty$, for any fixed k. It is clear that to obtain a non-flat limit, (namely

the original Schwarzschild metric), one must increase the descent level k in $\{x_j^k\}$ as a function of j, with $k = k(j) \to \infty$ as $j \to \infty$. One obtains a non-flat limit when the base points x_i^k satisfy $u(x_i^k) = 1 - \delta$, for any fixed $\delta > 0$. In this particular example, even though $k(j) \to \infty$ as $j \to \infty$, the potential function $u_j^{k(j)}$ does not converge to the 0-function as $j \to \infty$.

We now return to the main issue of constructing a (ρ, c) buffered sequence from some initial sequence whose blow-up limits are flat, with constant potential function.

Theorem 3.10 (Existence of Non-Flat Blow-ups). *Let $\{g_i\}$ be a sequence of unit volume Yamabe metrics on a closed 3-manifold M, satisfying the non-collapse assumption (3.43) and scalar curvature bound (3.41), i.e.*

$$-\infty < -s_o \leq s_{g_i} \leq -s_1 < 0. \tag{3.59}$$

Suppose there are points $x_i \in (M, g_i)$ and a number $\delta_o > 0$ such that

$$\rho(x_i) \to 0, \quad \text{as } i \to \infty, \quad \text{and} \quad u_i(x_i)/T_i \geq \delta_o > 0, \tag{3.60}$$

where $T_i \equiv \max_M u_i$ satisfies (3.45), i.e. for some $\lambda_o > 0$,

$$T_i \geq \lambda_o. \tag{3.61}$$

Then for any fixed $c > 0$ sufficiently small, there exists a sequence of (ρ, c)-buffered base points $\{y_i\}$, satisfying the following conclusions: $u_i(y_i) \in (0, T_i), \mathrm{dist}_{g_i}(x_i, y_i) \to 0$, and the rescalings

$$g_i' = \rho_i^{-2}(y_i) \cdot g_i, \quad \rho_i(y_i) \to 0, \tag{3.62}$$

of $\{g_i\}$ sub-converge, in the weak $L^{2,2}$ topology based at $\{y_i\}$ on a domain D, to a limit smooth metric g' on D, based at $y' = \lim y_i$. The pair (D, g') is a non-flat, (and non-super-trivial), solution to the static vacuum Einstein equations, with limit potential function $\bar{u}$ constructed from $\{u_i\}$. The limit domain D is naturally embedded in M via the metric g' and the convergence is in the strong $L^{2,2}$ topology away from the locus $\{\bar{u} = 0\}$.

Proof. The proof proceeds in several steps, organized around several lemmas. Recall again that the estimate (3.61) follows from a bound on the L^2 norm of z^T under the bounds (3.59), c.f. (3.45).

First, as previously in Proposition 3.1 and Remark 3.9 for example, it is necessary to renormalize u_i to $\bar{u}_i = u_i/T_i$, at least if $T_i \to \infty$. We will assume this is done, so that $\sup \bar{u}_i = 1$, and neglect the bar notation on $\bar{u}_i$.

Step I. To begin, the sequence $\{x_i\}$ satisfying (3.60) must be altered to a new sequence of points $\{x_i^1\}$, still satisfying (3.60), but with (locally) near-maximal u-values, (compare with (3.47)).

Lemma 3.11. (Initial Degeneration Points). *Under the assumption (3.60), there exists a subsequence $\{i'\}$ of $\{i\}$, relabeled to $\{i\}$, and base points x_i^1, with $\mathrm{dist}_{g_i}(x_i, x_i^1) \to 0$ which still satisfy (3.60), such that for any point $q_i \in B_{x_i^1}(\rho(x_i^1))$,*

$$u_i(q_i) \leq u_i(x_i^1) + \varepsilon_i, \tag{3.63}$$

for some sequence $\varepsilon_i = \varepsilon(x_i^1) \to 0$ as $i \to \infty$.

Proof. The proof is based on the fact that ρ is a Lipschitz function with Lipschitz constant 1, c.f. (1.27). Thus given a sequence x_i satisfying (3.60), suppose (3.63) does not hold in $B_{x_i}(\rho(x_i))$. Then there exist $q_i \in B_{x_i}(\rho(x_i))$ such that $u(q_i) \geq u(x_i) + \varepsilon_o$, for some $\varepsilon_o > 0$. But $\rho(q_i) \leq 2\rho(x_i) \to 0$ as $i \to \infty$, so that $\{q_i\}$ also satisfies (3.60). Clearly this procedure may be repeated any given number of times. Since $\max u_i = 1$, it follows that for any given $\varepsilon_o > 0$, there exist $x_i^{\varepsilon_o}$ such that, for all i sufficiently large, (3.63) holds with ε_o in place of ε_i and $\rho(x_i^{\varepsilon_o}) \to 0$ as $i \to \infty$. Now replace ε_o by a sequence $\varepsilon_j \to 0$ and choose a suitable diagonal subsequence j_i of $\{i, j\}$ with $\varepsilon_{j_i} \to 0$ sufficiently slowly as $i \to \infty$. Then the base points $x_i^1 \equiv x_i^{\varepsilon_{j_i}}$ satisfy (3.60) and (3.63).

■

From now on, we work with the subsequence $\{x_i^1\}$ from Lemma 3.11. Of course the choice of $\{x_i^1\}$ may be far from unique. Even more, sequences satisfying (3.60) may also be far from unique. The initial points x_i^1 should not be confused with the points in (3.47), whose existence was assumed and not proved. Lemma 3.11 is the only place where the hypothesis (3.60) is used in the proof of Theorem 3.10.

Now, as in (3.48), consider the rescaled metrics

$$g_i^1 = \rho(x_i^1)^{-2} \cdot g_i. \tag{3.64}$$

Given the hypotheses (3.59) and (3.61), the discussion preceding Proposition 3.6 shows that a subsequence of $\{g_i^1\}$ converges on $B^1_{x_i^1}(1)$ to a limit, which is a non super-trivial solution $(B^1(1), g^1, u^1, x^1)$ of the static vacuum equations. (With the exception of the non-collapse hypothesis (3.43), this is the only place where the remaining hypotheses, (3.59) and (3.61), are used in the proof of Theorem 3.10). For simplicity, we restrict ourselves in the following to this convergent subsequence. From (3.63), one sees as in Proposition 3.6, c.f.(3.53), that the limit is flat, with $u^1 = const$. If necessary, we renormalize $\{u_i\}$ again from now on so that $u^1 \equiv 1$ in the limit.

In the following steps, we show that for each i sufficiently large, there is a first $k_o = k_o(i)$ and points $x_i^{k_o} \in U^{k_o}$ such that the sequence of points $x_i^{k_o}$ are (ρ, c) buffered in the sequence (M, g_i), for a specified small $c > 0$. Here U^k and L^k are defined as preceding Remark 3.9. This is a quantitative version of the argument in Remark 3.9, and rests basically on the fact that (M, g_i, u_i) is smooth, for each fixed i.

The construction of such points will take place with a choice for the parameter c for a (ρ, c) buffered sequence, and a choice for a parameter ε measuring the distance of the blow-up metrics to a static vacuum solution. As will be seen in the proof, the choice of c and ε is arbitrary, provided each is sufficiently small, and further ε is sufficiently small, depending on c. While it is possible to make specific numerical choices for c and ε, the work involved is cumbersome and of no specific value, and so we will not do so. The final specification of c and ε will be made in Step IV below, after a preliminary specification in Step II.

From the discussion following (3.64), given an $\varepsilon > 0$, there is an $i_o = i_o(\varepsilon)$ such that for all $i \geq i_o$, the blow up $(B^1_{x_i^1}(1), g_i^1, x_i^1)$ is ε-close to a flat static vacuum limit. By ε-close, we mean that the L^2 norm of $L^* u_i$ on $B^1_{x_i^1}(1)$ is less than ε. Since u_i is C^o (in fact $L^{2,2}$) close to 1 at points in $B^1_{x_i^1}(1)$ an arbitrary fixed distance away from the boundary, this implies that the L^2 norm of the curvature r is less than $\varepsilon' = \varepsilon'(\varepsilon)$ in balls an arbitrary but fixed distance away from the boundary. In particular, by Theorem 3.4, ρ^1 is very small, (depending on ε), somewhere near the boundary of

$B^1_{x^1_i}(1)$. (Further, by Theorem 3.3, some portion of the level set L^2 is very close to the boundary of $B^1_{x_i}(1)$.

Step II. For any given $i \geq i_o = i_o(\varepsilon)$ fixed, and base points x^1_i as above, we now inductively construct new base points x^2_i, x^3_i, etc, chosen to satisfy quantitative versions of (3.56)-(3.57). For a given choice of c, $c_o = 10^{-3}$ and $\nu_o = 10^{-2}$, let $d = d(c, c_o, \nu_o)$ be the strong buffer constant determined by c, c_o and ν_o, as discussed following Lemma 3.8. For the moment, (until Step IV), we require only that c be chosen sufficiently small so that $d \leq \frac{1}{5}$ and $d + 2c \leq \frac{1}{4}$.

k^{th}**-Level Inductive Hypothesis:** For any given $i \geq i_o$, suppose base points $x^j_i \in U^j$, have been constructed for each j, $1 \leq j \leq k$, up to a given $k \geq 2$. We require that each x^j_i is not (ρ, c) buffered for $j \leq k - 1$, and to be chosen to satisfy the following conditions:

$$x^j_i \in B_{x^{j-1}_i}(\rho(x^{j-1}_i)), \tag{3.65}$$

$$c \cdot \rho(x^{j-1}_i) \leq \mathrm{dist}_{g_i}(x^j_i, \partial B_{x^{j-1}_i}(\rho(x^{j-1}_i))) \leq 2c \cdot \rho(x^{j-1}_i), \tag{3.66}$$

$$\rho(x^j_i) \leq d_1 \cdot \rho(x^{j-1}_i), d_1 = d + 2c, \quad \text{and} \tag{3.67}$$

$$|u_i(x) - u_i(x^{j-1}_i)| \leq \tfrac{1}{4} \cdot u_i(x^{j-1}_i), \qquad \forall x \in B_{x^{j-1}_i}((1-c)\rho(x^{j-1}_i)). \tag{3.68}$$

The conditions (3.65)-(3.66) on the placement of x^j_i, given x^{j-1}_i, can obviously always be satisfied. With regard to the next two conditions, condition (3.68) on the behavior of the potential, and the fact that x^{j-1}_i is not (ρ, c) buffered, would imply that x^{j-1}_i is not strongly (ρ, d) buffered provided Lemma 3.8 is applicable. If this is the case, it follows that there is a point $z^j_i \in B_{x^{j-1}_i}(\rho(x^{j-1}_i))$, with $\rho(z^j_i) \leq d \cdot \rho(x^{j-1}_i)$, and hence by (1.27), there is a point x^j_i satisfying (3.65)-(3.66) together with (3.67). In other words, (3.67) is a consequence of (3.65), (3.66) and (3.68) to the extent that Lemma 3.8 is applicable. We will show below that indeed it is, even when $u_i(x^{j-1}_i)$ is very small.

It is important to note that the terminal base point x^k_i may or may not be (ρ, c) buffered. Further, by the paragraph preceding Step II, there exist base points x^2_i satisfying (3.65)-(3.68) with $k = 2$ whenever $i \geq i_o$, so that one may start the induction process at $k = 2$.

To normalize the discussion, we scale the metrics g_i at each x^j_i so that $\rho^j_i = 1$, i.e. set $g^j_i = \rho(x^j_i)^{-2} \cdot g_i$, and renormalize u_i by setting

$$u^j_i = u_i / u(x^j_i). \tag{3.69}$$

With this normalization, (3.65)-(3.68) are equivalent to the statements

$$x^j_i \in B^{j-1}_{x^{j-1}_i}(1), \tag{3.70}$$

$$c \leq \mathrm{dist}_{g^{j-1}_i}(x^j_i, \partial B^{j-1}_{x^{j-1}_i}(1)) \leq 2c, \tag{3.71}$$

$$\rho^{j-1}(x^j_i) \leq d_1, \quad \text{and,} \tag{3.72}$$

$$|u^{j-1}_i(x) - 1| \leq \tfrac{1}{4}, \qquad \forall x \in B^{j-1}_{x^{j-1}_i}(1-c). \tag{3.73}$$

The inductive step is the following:

Inductive Step. Suppose x_i^k is not (ρ, c) buffered. Then there exist base points x_i^{k+1} satisfying the conditions (3.65)-(3.68), with $k+1$ in place of k.

The next two steps are concerned with the proof of the inductive step.

Step III. In order to carry out the inductive step from k to $k+1$, we need to analyse the geometry of g_i^k in the unit ball $B_{x_i^k}^k(1)$. Note that the following Lemma does not assume any buffer condition on x_i^k, as a sequence in i, for any k fixed.

Lemma 3.12. *For any $i \geq i_o$ and some $k \geq 2$, suppose base points x_i^k have been constructed satisfying the k^{th}-level inductive hypothesis. Then the triple $(B_{x_i^k}^k(1), g_i^k, u_i^k)$ satisfies the estimates*

$$u_i^k r_i = D^2 u_i^k + o(i,k), \qquad \Delta u_i^k = o(i,k), \tag{3.74}$$

where $o(i,k)$ denotes terms which become arbitrarily small in $L^2(B_{x_i^k}^k(1))$, (in fact in $L^2(M, g_i^k)$), if either i or k is sufficiently large.

Proof. We return to the equation (2.10) for $u = u_i$, i.e.

$$ur - D^2 u = -\Delta u \cdot g + \xi + \frac{s}{3} \cdot g,$$

applied to g_i^k in B_i^k. Divide, i.e. renormalize, this equation by $\upsilon_{i,k} \equiv u_i(x_i^k)$. By (3.66) and (3.68), $\upsilon_{i,k} \geq 2^{-k}$. Let $u_i^k = u_i/\upsilon_{i,k}$ as in (3.69), so that $u_i^k(x_i^k) = 1$. This gives

$$u_i^k r - D^2 u_i^k = -\Delta u_i^k \cdot g + \upsilon_{i,k}^{-1} \cdot \xi + \upsilon_{i,k}^{-1} \cdot \frac{s}{3} \cdot g. \tag{3.75}$$

Now we claim that the term $\upsilon_{i,k}^{-1} \cdot \xi$ is arbitrarily small in L^2, if either i or k is sufficiently large. To see this, recall from Theorem 2.2 that $||\xi||_{L^2(M,g_i)} \leq C$. As noted before in (3.12), in any rescaling $g_i' = \rho_i^{-2} \cdot g_i$, one has then

$$||\xi||_{L^2(M,g_i')} \leq C \cdot \rho_i^{1/2}. \tag{3.76}$$

Now by (3.72), in the sequence g_i^j, for i fixed and any $j < k$, we have

$$\rho_i^j \leq d_1 \cdot \rho_i^{j-1},$$

so that

$$\rho_i^k \leq d_1^k \cdot \rho_i^1.$$

It follows that

$$||\upsilon_{i,k}^{-1} \xi||_{L^2(M,g_i^k)} \leq C \cdot 2^k \cdot d_1^{k/2} \rho(x_i^1)^{1/2} \to 0,$$

as either i or $k \to \infty$, since $d_1 = d + 2c \leq 1/4$. The same reasoning applies to the last term in (3.75) involving the scalar curvature, since the scalar curvature of (M, g_i) is uniformly bounded in L^∞. Thus, the last two terms in (3.75) are arbitrarily small, if either i or k is sufficiently large.

Similarly, from Proposition 2.9, the L^2 norm of Δu is uniformly bounded on (M, g_i) and hence again the same reasoning implies that the L^2 norm of Δu_i^k goes to 0 uniformly on (M, g_i^k), as either $i \to \infty$ or $k \to \infty$. Thus the right hand side (3.75) is arbitrarily small whenever either i or k is sufficiently large.

■

Lemma 3.12, together with the non-collapse assumption (3.43) implies that for any $i \geq i_o = i_o(\varepsilon)$ and $k \geq 2$, the metrics g_i^k are ε-close to a static vacuum solution in the sense that the metrics satisfy the static vacuum equations with a (volume normalized) L^2 error of at most ε. Observe that this ε is the same as the ε chosen initially at the end of Step I. By passing to limits, it follows that in fact g_i^k on B_i^k is $\varepsilon' = \varepsilon'(\varepsilon)$ close in the weak $L^{2,2}$ topology to an actual static vacuum solution; however, we will not use this fact. By the inductive hypothesis, the predecessors g_i^j, $j \leq k-1$, are not (ρ, c) buffered at their base points, and so are all δ_1-close to a flat metric in $B_i^j(1-c)$ in the sense that

$$\Big(\int_{B_i^j(1-c)} |r|^2\Big)^{1/2} \leq \delta_1; \tag{3.77}$$

here $\delta_1 = \delta_1(c, c_o) = 2\pi c \cdot c_o$. (Again it can be shown that this implies that $(B_i^j(1-c), g_i^j)$ is δ_1'-close, $\delta_1' = \delta_1'(\delta_1, \nu_o)$, in the weak $L^{2,2}$ topology, to a flat metric, but this will not be used). Clearly, δ_1 may be made arbitrarily small by choosing the initial buffer parameter c sufficiently small. Further, by (3.73), observe that the potential function u_i^j is $\frac{1}{4}$-close to a constant function in the C^0 topology in $B_i^j(1-c)$.

Step IV. We are now in position to prove the inductive step. The next result is the main point in this respect. For this, we restrict somewhat further the initial choice of c, and ε. Thus, we require for the remainder of the proof that $\varepsilon < \varepsilon_o$, where ε_o is the constant from Lemma 3.8, and also $\varepsilon < \frac{1}{10}\delta_1$. In addition, c will be chosen sufficiently small so that $d_1 = d + 2c \leq \frac{1}{100}$, and $\delta_2 = 2c_s \cdot \delta_1 \leq \frac{1}{100}$, where c_s is the Sobolev constant of the embedding $L^{2,2} \subset C^0$ in $\mathbb{R}^3$, (c.f. (3.79) below).

Lemma 3.13. *Under the conditions above on c and ε, for any given $i \geq i_o = i_o(\varepsilon)$, suppose the base point x_i^k is not (ρ, c) buffered. Assume also that the non-collapse assumption (3.43) holds. Then the potential function u_i^k is $\frac{1}{4}$-close to the constant function 1 on $B_{x_i^k}^k(1-c)$ in the C^0 topology, i.e. (3.73) holds with $k+1$ in place of k.*

Proof. The proof is essentially the same as the proof of the similar issue in Remark 3.9, c.f. (3.58). Since x_i^k is not (ρ, c) buffered and (3.43) holds, the metric g_i^k is both ε-close to a static vacuum solution, and $\delta_1 = \delta_1(c)$ close to a flat metric, in the sense defined above, on $B^k(1-c) \equiv B_{x_i^k}^k(1-c)$.

Let $v_i^k = sup_{B^k(1-c)}|u_i^k|$, and recall that $u_i^k(x_i^k) = 1$. It follows from Lemma 3.12 that

$$\Big(\int_{B^k(1-c)} |D^2 u_i^k|^2\Big)^{1/2} \leq \Big(\int_{B^k(1-c)} (u_i^k)^2 |r|^2\Big)^{1/2} + \varepsilon \leq 2\delta_1 \cdot v_i^k. \tag{3.78}$$

This, together with Sobolev embedding, c.f. [GT, Ch.7], shows that the potential u_i^k is close in the C^0 topology to an affine function $\alpha = \alpha_{i,k}$ on $B^k(1-c)$, in that

$$|u_i^k - \alpha| \leq \delta_2 v_i^k. \tag{3.79}$$

The constant δ_2 is given by $\delta_2 = 2c_s \cdot \delta_1 \leq \frac{1}{100}$, where c_s is the Sobolev embedding constant; c_s is a fixed numerical constant in dimension 3, independent of all parameters. This estimate also, of course, applies on domains inside $B^k(1-c)$, with the same α and with v_i^k replaced by the supremum of $|u_i^k|$ on the domain. In particular, it follows that $sup|u_i^k|$ is close to $sup|\alpha|$ on any such domain.

Now to understand more precisely the form of α and the size of v_i^k, we first consider the validity of (3.79) on the predecessor domain, in the g_i^k scale. Thus, consider the previous point x_i^{k-1}, of distance R_i^k from x_i^k in the g_i^k scale. By (3.71)-(3.72), one computes $d_1^{-1}(1-2c) \leq R_i^k \leq c^{-1}(1-c)$, the important point being that R_i^k is large, since d_1 is small. Consider also the ball $B^{k-1}(R_i^k) \equiv B^k_{x_i^{k-1}}(R_i^k)$ containing a portion (roughly half) of the ball $B^k(1-c)$. Using the inductive assumption, the L^2 norm of the curvature of g_i^k on $B^{k-1}(R_i^k)$ is much smaller than δ_1, since g_i^k is much larger than g_i^{k-1}. This implies that (3.79) holds on $B^{k-1}(R_i^k)$, i.e.

$$|u_i^k - \alpha'| \leq \delta_2 w_i^k, \tag{3.80}$$

where α' is an affine function on $B^{k-1}(R_i^k)$ and w_i^k is the supremum of $|u_i^k|$ on $B^{k-1}(R_i^k)$. We may write $\alpha' = a' + b't$, where t is an affine Euclidean coordinate with $t(x_i^k) = 0$.

By the induction hypothesis (3.73), the function

$$u_i^k = \frac{u_i(x_i^{k-1})}{u_i(x_i^k)} u_i^{k-1}$$

is $2 \cdot \frac{1}{4}$ close to the constant function 1 in $B^{k-1}(R_i^k)$, and hence $w_i^k \leq \frac{3}{2}$. Together with (3.80), this implies that $\alpha' = a' + b't$ satisfies $|b'| \leq (R_i^k)^{-1} < 2d_1$. Thus the oscillation of α' is small. Further $|1 - \alpha'| \leq 2\delta_2$.

It follows that on the 'half-ball' $D_i^k \equiv B^k(1-c) \cap B^{k-1}(R_i^k)$, the function u_i^k satisfies

$$|u_i^k - 1| \leq 2(\delta_2 + d_1). \tag{3.81}$$

Now return to the estimate (3.79). It follows from (3.81) that $sup_{D^k}|u_i^k|$ is close to 1. Hence, by (3.79) and the discussion in the paragraph following it, one may calculate that $v_i^k = sup_{B^k(1-c)}|u_i^k| \leq 5$. It then follows from the estimates (3.78)-(3.79) applied to the union $B^k(1-c) \cup B^{k-1}(R_i^k)$ that α and α' are close. Working out concretely the degree of closeness then gives the estimate

$$|u_i^k - 1| \leq 10(\delta_2 + d_1), \tag{3.82}$$

on $B^k(1-c)$. The result then follows from the specifications of δ_2 and d_1 above.

∎

We are now in position to argue the inductive step.

Lemma 3.14. *Suppose x_i^k satisfies the k^{th}-level inductive hypothesis (3.65)-(3.68). If the base point x_i^k is not (ρ, c) buffered, then there exist points x_i^{k+1} satisfying the $(k+1)^{\text{st}}$-level inductive hypothesis, i.e. (3.65)-(3.68) with $k+1$ in place of k everywhere.*

Proof. Given the previous work, this is now essentially obvious. There are clearly (many) base points x_i^{k+1} satisfying (3.65)-(3.66) with $k+1$ in place of k. Since x_i^k is not (ρ, c) buffered, by Lemma 3.13 u_i^k is $\frac{1}{4}$-close to the constant function 1 in $B^k_{x_i^k}(1-c)$, so that (3.68), (or (3.73)) holds, with $k+1$ in place of k. By Lemma 3.12, or more precisely its contrapositive, x_i^k is not strongly (ρ, d) buffered, and so there exist x_i^{k+1} such that (3.67) also holds, with $k+1$ in place of k.

∎

Step V. Now, since (M, g_i) is a smooth compact Riemannian manifold, $\rho(M, g_i)$ cannot be arbitrarily small for any given $i \geq i_o$ i.e. there exists $\rho_o = \rho_o(i)$ such that $\rho(M, g_i) \geq \rho_o > 0$. Starting with the initial sequence x_i^1 from Lemma 3.11, and any $i \geq i_o$, the inductive process above constructs points $x_i^k, k \geq 2$, satisfying $\rho(x_i^k) \leq d_1\rho(x_i^{k-1})$, and continues indefinitely as long as x_i^k is not (ρ, c) buffered. It follows that there is a first level $k_o = k_o(i)$ such that $y_i \equiv x_i^{k_o}$ is (ρ, c) buffered. Observe that the discussion in Remark 3.9 proves that necessarily $k_o(i) \to \infty$ as $i \to \infty$. The sequence $(B_i^{k_o}(1), g_i^{k_o}, y_i, u_i^{k_o})$ satisfies the equation (3.75), with the right side going to 0 uniformly in $L^2(M, g_i^{k_o})$ as $i \to \infty$. Thus, (c.f. the discussion following Definition 3.7), a subsequence converges to a non-flat limit solution (D, g') to the static vacuum equations, with (possibly renormalized) potential function $\bar{u} = \lim u_i^{k_o}$. (Here we are using again the non-collapse assumption (3.43)). The limit domain D may be chosen for instance to be an open domain in the limit $B_{y'}(1)$ of $B_i^{k_o}(1)$ which has smooth and compact closure in $B_{y'}(1)$. This completes the proof of Theorem 3.10.

■

We conclude this subsection with a number of remarks.

Remark 3.15. (i). At this point, if not well before, one may wonder why not just blow-up the sequence $\{g_i\}$ at base points $\{q_i\}$ realizing the minimum $\rho(M)$ of $\rho(x), x \in \{(M, g_i)\}$, in which case $\{q_i\}$ is strongly $(\rho, 1)$ buffered. In this case, a limit of the metrics

$$g_i' = \rho(M)^{-2} \cdot g_i$$

will automatically be complete and at least $L^{2,2}$ smooth.

The main problem is that, from the work above, there is every indication that this minimum occurs at, or arbitrarily near the 0-levels of u. The z-splitting of $\{g_i'\}$ has the form

$$u_i z_i' = (D^2)' u_i - \Delta_i' u_i + \xi_i' - \frac{s_i'}{3} \cdot g_i'.$$

Although, as before, the last three terms on the right go to 0, it may well happen that u_i converges to 0 uniformly when blowing up at $\{q_i\}$, so that in the limit this equation gives just $0 = 0$, i.e. a super-trivial solution, and thus no information whatsoever regarding the limit metric. Examples exhibiting exactly this behavior will be discussed in §6. Further, without the descent construction as in the proof of Theorem 3.10, there may be no means to renormalize u to obtain a non-trivial solution.

(ii). It is clear that all the preceding arguments are unchanged if u is replaced by $-u$, so that the construction of Theorem 3.10 may be carried out either “down” the u-levels from a local maximum, as in (3.63), or “up” the u-levels from a local minimum, provided such a local minimum is bounded away from zero.

(iii). We point out that it is not possible in the construction above to descend down to other levels besides the 0-level, e.g. trying to descend only to the level $u = \frac{1}{2}$ by renormalizing the differences $\frac{1}{2} + 2^{-k-1}$ and $\frac{1}{2} + 2^{-k}$. This is due essentially to the linear, and not affine, nature of the equation (3.71) in u.

(iv). The construction in Theorem 3.10 has been referred to as a ‘descent’ construction down the levels of $u = u_i$, since this is what occurs in all known examples, (as in Example 3.9), and since one certainly descends to u-levels less than $\max u_i/T_i = 1$. However, the construction itself does

not guarantee such a descent, i.e. that

$$u_i(x_i^j) < u_i(x_i^{j-1}) \qquad \text{for } j \leq k_o. \tag{3.83}$$

Only the control (3.68) or (3.73) is obtained on the behavior of u.

On the other hand, although it has not been used in the proof, it is worth noting that at any stage in the inductive construction, the level sets L^s of u_i^j, $j < k_o$, for any given s with $0 < s < 10^{-1}$, are somewhere $\delta = \delta(d)$ close to $\partial B_{x_i^j}^j(1)$ in the g_i^j metric; here δ may made arbitrarily small by choosing $d = d(c)$ sufficiently small. This follows from the fact that x_i^j is not (ρ, c) buffered, and hence not strongly (ρ, d) buffered, by Lemma 3.8.

Using this remark, it is possible to define variants of the inductive construction in Theorem 3.10 for which (3.83) does hold, and for which the same conclusions are valid. Such variations will not be used here however.

Remark 3.16. (i). The construction in Theorem 3.10 is very local. Thus, given any initial sequence $\{x_i^1\}$ satisfying the conclusions of Lemma 3.11, there is a buffered sequence $\{y_i\}$ satisfying the conclusions of Theorem 3.10 very close to x_i^1; in fact $\mathrm{dist}_{g_i}(x_i^1, y_i) \leq 2\rho(x_i^1)$.

However, there may be many possible (essentially) distinct choices of the initial sequence $\{x_i^1\}$ in (M, g_i) satisfying (3.60) and (3.63). Note also that the choice of y_i is by no means unique, since the choice of the intermediate points x_i^j may allow considerable freedom.

(ii). The limit solution (D, g') in Theorem 3.10 is only locally defined, and further may live in the region of M where u is approximately 0, since the potential function $\bar{u}$ is obtained by renormalizing, possibly infinitely many times, the original sequence $\{u_i\}$. The geometry of the limit solutions (D, g') will be studied in more detail in §5.

Remark 3.17. The proof of Theorem 3.10 requires Lemma 3.11 to obtain the initial sequence $\{x_i^1\}$ whose blow-up limit is a flat static vacuum solution, with limit potential function $u^1 = \mathit{const.} > 0$. On the other hand, if one is given some initial sequence $\{p_i^1\}$ satisfying this latter property, (not necessarily satisfying the local maximum condition (3.63)), then the construction in Theorem 3.10 may be applied starting at $\{p_i^1\}$, without any further changes, to reach the same conclusions.

There are instances where the hypothesis (3.60) in Theorem 3.10 can be deduced or replaced by suitable assumptions on the local behavior of the functions u_i. We give two examples of this below.

Proposition 3.18. *Let (M, g_i) be a sequence of unit volume Yamabe metrics satisfying the non-collapse assumption (3.43). Suppose there exists a $\delta > 0$, a sequence $r_i \to 0$, and a sequence of base points $\{x_i\} \in (M, g_i)$, such that on $B_i = B_{x_i}(r_i)$,*

$$\operatorname*{osc}_{B_i}(u_i/T_i) \geq \delta. \tag{3.84}$$

Then (3.60) holds, with δ in place of δ_o.

Proof. It suffices to show that $\rho(x_i) \to 0$ as $i \to \infty$. If there exists $\rho_o > 0$ such that $\rho(x_i) \geq \rho_o$, (in a subsequence), then L^2 estimates on the trace equation (2.13), c.f. [GT, Thm.8.8], imply that $\{u_i/T_i\}$ is uniformly bounded in $L^{2,2}(B_{x_i}(\rho_o/2))$. Sobolev embedding then implies that u_i/T_i is uniformly bounded in $C^{1/2}$, so that in particular the oscillation of u_i/T_i on $B_{x_i}(r)$ is bounded by

$r^{1/2}$. This contradicts (3.84).

■

Somewhat more generally, the hypotheses (3.60) and (3.61) in Theorem 3.10 can be replaced by the following local condition on the behavior of $u = u_i$ near its 0-level set. Let $t(x) = \mathrm{dist}_{g_i}(x, L^o)$. If $u_i > 0$ everywhere, L^o must be replaced by L^{ε_i}, for a suitable sequence $\varepsilon_i \to 0$.

Proposition 3.19. *Let (M, g_i) be a sequence of unit volume Yamabe metrics satisfying (3.59) and the non-collapse assumption (3.43). Suppose there exists a sequence of base points $\{x_i\} \in (M, g_i)$, with $t(x_i) \to 0$ as $i \to \infty$, such that*

$$u_i(x_i) \gg t^{1/2} \qquad \text{as } i \to \infty, \tag{3.85}$$

and

$$t|\nabla log u_i|(q_i) < \tfrac{1}{2}, \tag{3.86}$$

for all $q_i \in B_{x_i}(K_i t(x_i))$ satisfying $\frac{1}{2}t(x_i) \leq t(q_i) \leq K_i t(x_i)$; here K_i is any given sequence with $K_i \to \infty$ as $i \to \infty$ and it is assumed that t achieves the value $K_i/2$ in $B_{x_i}(K_i t(x_i))$. Then the conclusions of Theorem 3.10 hold for some (ρ, c) buffered sequence y_i with $dist_{g_i}(x_i, y_i) \to 0$ as $i \to \infty$.

Proof. First, note that as in Proposition 3.18, (3.85) forces $\rho(x_i) \to 0$, since if $\rho(x_i)$ were bounded away from 0, L^2 elliptic estimates applied to the trace equation (2.13) as above imply that u_i is bounded in $L^{2,2} \subset C^{1/2}$, which violates (3.85).

Consider the behavior of the pointed sequence $(M, g_i', x_i, \bar{u}_i), g_i' = t(x_i)^{-2} \cdot g_i$. We claim that Lemma 3.12 holds on $B_{x_i}'(1)$. Thus, for $\bar{u}_i = u_i/u(x_i)$, (compare with (3.75)), the renormalized equation for u takes the form

$$\bar{u}_i r - D^2\bar{u}_i = -\Delta\bar{u}_i \cdot g + u(x_i)^{-1} \cdot \xi + u(x_i)^{-1} \cdot \frac{s}{3} \cdot g. \tag{3.87}$$

As in (3.76), in this scale, we have

$$u(x_i)^{-1}||\xi||_{L^2} \leq C \cdot u(x_i)^{-1} t(x_i)^{1/2} \to 0, \quad \text{as } i \to \infty,$$

where the last estimate follows from (3.85). The same reasoning as in Lemma 3.12 shows that the full right side of (3.87) also goes to 0, as $i \to \infty$, which proves the claim.

Next we claim that there exists a constant $\rho_o > 0$ such that

$$\rho(x_i) \geq \rho_o \cdot t(x_i). \tag{3.88}$$

For in the scale g_i' where $t'(x_i) = 1$, note that $\bar{u}_i(x_i) = 1$, and by (3.86), $\bar{u}_i$ is bounded away from 0 in the region where $\frac{1}{2} \leq t' \leq 2$; of course here $t'(p) = \mathrm{dist}_{g_i'}(p, L^o)$. Hence, (3.88) follows from Theorem 3.3 applied to the triple $(M, g_i', \bar{u}_i)$. Since $\rho'(x_i)$ is bounded away from 0, blow-up limits based at x_i exist. Further, the preceding discussion shows that all blow-up limits based at x_i converge to a static vacuum solution with potential function $\bar{u}$.

Now if a blow-up limit (M, g_i', x_i) happens to be non-flat, we are done (the assertion is proved in this case). Suppose instead that the blow-up limit is flat. Then the limit function $\bar{u}$ is either a non-constant affine function or $\bar{u} = const. = 1$. Hence it suffices to prove that the latter case holds, since then all the arguments following Lemma 3.11 can be carried over without change, c.f. Remark 3.17.

Suppose the limit $\bar{u}$ is a non-constant affine function. The product (3.86) is scale-invariant, and hence on the limit, one obtains

$$t'|\nabla log\bar{u}|(q) \leq \tfrac{1}{2},$$

for all q in $B'_{x_i}(K_i)$ satisfying $t'(q) \geq \frac{1}{2}$. The assumption on t and K_i implies that on the limit, t' is an unbounded function. However, it is obvious that no non-constant affine function satisfies this bound.

■

To conclude this subsection, it is clear that Theorem 3.10 proves Theorem A(II) in case $\sigma(M) < 0$, or more precisely in case the scalar curvature of $\{g_i\}$ satisfies (3.59).

In the following two subsections, we complete the proof of Theorem A(II) and Theorem 3.10 in the cases $\sigma(M) = 0$ and $\sigma(M) > 0$. As will be seen, this requires some, but as it turns out, no truly essential changes.

§3.5. Suppose

$$\sigma(M) = 0,$$

or more precisely suppose $\{g_i\}$ is a sequence of unit volume Yamabe metrics on M satisfying

$$s_{g_i} \to 0. \tag{3.89}$$

As before, we require that the assumptions (3.42) and (3.43) hold uniformly on $\{g_i\}$.

Of course, if λ does not converge to 0, so that (3.45) holds, then all of the results of §3.4 still hold, with identical proofs, at least when $s_{g_i} \leq 0$. However, as discussed in Remark 2.12, one often expects that $\lambda \to 0$ when (3.89) holds. In fact, if $\sigma(M) > 0$ and $s_{g_i} \to 0$, then as discussed following Proposition 2.9, there are many unit volume Yamabe metrics g_i which do not degenerate at all, and for which $f \to -1,\ u \to 0$, and $\xi \to 0$ smoothly.

Thus, suppose that $\lambda = \lambda_i \to 0$ on (M, g_i). Assuming $\{g_i\}$ degenerates in the sense of (0.8), since u may go to 0 in L^2, one might not expect to produce any non-trivial blow-up solutions of the static vacuum equations by descent down the u-levels. However, when $\lambda \to 0$, we also have $\delta = -\int k \to 1$, by (2.17) and (2.28). In particular, $\sup(-k)$ is uniformly bounded away from 0 on $\{g_i\}$, compare with (3.61).

Given these preliminaries, we prove Theorem 3.10 with the assumption (3.89) replacing (3.59).

Proof of Theorem 3.10: $\sigma(M) = 0$**.**

We assume that $\{g_i\}$ is a degenerating sequence of unit volume Yamabe metrics on $U_{\delta_o} \subset M$, satisfying (3.89), together with the L^2 bound on z^T (3.42), (or (0.19)), and the non-collapse assumption (3.43). For simplicity, assume first that

$$s_{g_i} \leq 0. \tag{3.90}$$

The splitting equation (2.25) for z^T may be written in the form

$$kr = D^2k - \Delta k \cdot g - z^T + \xi. \tag{3.91}$$

By (3.42), (and also (2.11)), the term $z^T - \xi$ is uniformly bounded in L^2, as is the term Δk, (by taking the trace of (2.25)).

This equation, and its associated trace equation, now have exactly the same form and bounds as the u-equation (2.10), with $-k$ in place of u and $z^T - \xi$ in place of $\xi + \frac{s}{3}g$. As noted above, the mean value of $-k$, namely δ, converges to 1 as $i \to \infty$.

Exactly the same analysis as in §3.3 and §3.4 may now be carried out on the function $-k$, or more precisely $-k/\sup(-k)$ in place of $u/\sup u$. Note that from (2.28),

$$\frac{-k}{\sup(-k)} = \frac{u}{\sup u}, \tag{3.92}$$

so that the main hypothesis (3.60) holds with $-k$ in place of u. Similarly, analogous to (3.61), $\sup(-k)$ is uniformly bounded away from 0, since its mean value converges to 1. Thus, one produces a (ρ, c) buffered sequence with associated limit giving a non-flat static vacuum solution, with limit potential function $\bar{k}$ coming from the geometry of $\{k_i\}$. In fact, by (3.92), the potential function can also be obtained from renormalizations of the $\{u_i\}$.

The proof in case

$$s_{g_i} \geq 0, \quad \text{and} \quad s_{g_i} \to 0, \tag{3.93}$$

is exactly the same as in the case (3.90) with one exception. Namely, if $s_{g_i} > 0$, it is possible that Ker $L^* \neq 0$ on (M, g_i) so that the functions u or k may not be uniquely defined. In this situation, one just takes any choice for $-k$ in $\{-k + \text{Ker } L^*\}$. Since the choice is always renormalized by its supremum, this has no effect on any of the arguments. This situation will be discussed in more detail in §3.6, where it occurs more naturally.

This completes the proof of Theorem 3.10 and Theorem A in case $\sigma(M) = 0$, i.e. (3.89) holds. $\blacksquare$

§3.6. Finally, we turn to the case where

$$\sigma(M) > 0,$$

or more generally where $\{g_i\}$ is a sequence of unit volume Yamabe metrics with

$$s_{g_i} \geq s_o > 0. \tag{3.94}$$

As noted in §2, such metrics may satisfy

$$\text{Ker } L^* \neq 0. \tag{3.95}$$

For instance, on the standard sphere S^3, the 1st eigenfunctions of the Laplacian form exactly Ker L^*, see also §6.4. Note that the condition (3.95) implies, by taking the trace, that

$$-s/2 \in \operatorname{Spec} \Delta. \tag{3.96}$$

The condition (3.95) is equivalent to the statement that the linearization L of s at a Yamabe metric g is not surjective onto $C^\infty(M, \mathbb{R})$, and has been studied by several authors, c.f. [Bo], [Kb2], [La]. In particular, the last two authors show that there are many (even conformally flat) Yamabe metrics satisfying (3.95) which are not Einstein.

Nevertheless, even for metrics satisfying (3.95), all the splittings (2.6), (2.10) (2.21) and (2.25) are valid and defined as before. The associated functions f, u and k however are obviously not uniquely defined; they are unique only modulo Ker L^*. Related to this, as noted following Proposition 2.9, there is no apriori bound on the $T^{2,2}$ norm or even the L^2 norm of f.

On the other hand, if g is a Yamabe metric with $s_g > 0$, then by the definition of g - as realizing $\inf \mathcal{S}$ in the conformal class $[g]$ - one has

$$s_g \cdot \left(\int \psi^6 dV_g \right)^{1/3} \leq \int (8|d\psi|^2 + s\psi^2)\, dV_g, \tag{3.97}$$

for any positive smooth function ψ on M, cf. [Bes,4.28], [LP]. It follows that (3.97) holds for all smooth functions ψ on M. Thus on (M, g) one has a Sobolev inequality, uniform on any sequence of Yamabe metrics for which $s_g > 0$ is uniformly bounded away from 0.

This has two immediate consequences. First, we have:

Lemma 3.20. *Let (M, g_i) be a sequence of unit volume Yamabe metrics on M satisfying (3.94). Then there is a constant $\nu_o = \nu_o(s_o)$ such that the volume radius satisfies*

$$\nu_i(x) \geq \nu_o, \tag{3.98}$$

for all $x \in (M, g_i)$. Thus, the sequence $\{g_i\}$ cannot collapse anywhere, c.f. §1.4.

Proof. It is a standard fact that the Sobolev inequality (3.97) gives rise to a uniform lower bound on the volumes of geodesic balls in comparison to the volumes of Euclidean balls, i.e. to the ratio $\mathrm{vol}(B_x(r))/r^3$, provided $\mathrm{vol}\, B_x(r) < \frac{1}{2}\, \mathrm{vol}(M, g_i)$; we refer to [Ak] for example for a proof. By definition of the volume radius, this gives (3.98). Similarly, it follows from the definition of collapse in §1.4 that (3.98) prevents collapse of the sequence (M, g_i) at any sequence of base points.

■

Hence, in the case of uniformly positive scalar curvature, condition (i) of Theorem A is automatically satisfied.

Second, from the trace equation (2.12),

$$2\Delta f + sf = \mathrm{tr}\ \xi, \tag{3.99}$$

and the fact that the right side is uniformly bounded in L^2, see Theorem 2.2, it follows from standard elliptic estimates using only the Sobolev inequality (3.97), c.f. [GT, Thm.8.15], that

$$\sup |f| \leq c(s_o) \left\{ \left(\left(\int f^2\, dV_g \right)^{1/2} \right) + 1 \right\}. \tag{3.100}$$

Of course, the $T^{2,2}$ norm of f is also bounded by the L^2 norm of f; see the proof of Proposition 2.9.

We now complete the proof of Theorem 3.10, with (3.94) in place of (3.59).

Proof of Theorem 3.10: $\sigma(M) > 0$.

Assume that $\{g_i\}$ is a degenerating sequence of unit volume Yamabe metrics on $U_{\delta_o} \subset M$, satisfying (3.94), together with the L^2 bound on z^T (3.42), (or (0.19)). By Lemma 3.20, no non-collapse assumption is needed.

As noted in Remark 2.12, the bound (3.94) implies that $\lambda = \lambda_i$ is bounded away from 0, and hence $T_i = \sup u_i \geq \lambda_o$, for some $\lambda_o > 0$; thus one has the bound (3.61) in this context.

If Ker $L^* = 0$ for all g_i, then all of the previous arguments in §3, in particular the proof of Theorem 3.10, hold without any changes.

Suppose instead that Ker $L^* \neq 0$ on some subsequence of $\{g_i\}$. In this case, simply make some choice for the function $u = u_i \mod \text{Ker } L^*$. Note that $\lambda = \int u$ is independent of any choice. As indicated above in §3.5, recall that throughout the previous work in §3, the potential u is initially renormalized by its maximum, i.e. only the behavior of $u/\sup u$ is considered. By (3.100) this renormalization is equivalent to the renormalization of u by its L^2 norm. Of course if $||u_i||_{L^2} \to \infty$, then the initial structural equation (2.10) becomes

$$L^*\left(\frac{u_i}{||u_i||_{L^2}}\right) \to 0, \quad \text{in } L^2(M, g_i), \quad \text{as } i \to \infty,$$

which is even stronger than the initial L^2 bound on $L^*(u)$, compare with the renormalization in the proof of Proposition 3.1. The proofs of all the previous results in §3 follow exactly as before, without any changes.

This completes the proof of Theorem A and Theorem 3.10 in all cases.

■

Remark 3.21. We point out that the assumption that the metrics $\{g_i\}$ in Theorem A or Theorem 3.10 are defined on a fixed 3-manifold M has not been used. In fact, these and all previous results hold under the same assumptions on arbitrary sequences of closed oriented Riemannian 3-manifolds (M_i, g_i).

4. Remarks on the Hypotheses.

In this section, we make some further comments on the hypotheses of Theorem A. First, we consider the non-collapse hypothesis (i) in Theorem A, i.e. (0.18) or more generally (3.43), and afterwards the degeneration hypothesis (iii) in Theorem A, i.e. (0.20). The hypothesis (ii), i.e. the L^2 bound on z^T is already discussed in Theorem 2.10-Remark 2.12.

§4.1. By Lemma 3.20, the non-collapse assumption (3.43) is only needed in the case $\sigma(M) \leq 0$. From some perspectives however, this assumption is perhaps the most crucial in Theorem 3.10 or Theorem A.

The basic difficulty in handling the case of collapse as opposed to convergence is already seen in the discussion in §3.2, for instance in the proof of Proposition 3.1. Thus, consider a blow-up sequence $g_i' = \rho(x_i)^{-2} \cdot g_i$, with $\rho(x_i) \to 0$. If this sequence collapses, that is

$$\nu(x) << \rho(x), \tag{4.1}$$

then it can be proved that the collapse is along a sequence of injective F-structures on $B'_{x_i}(1-\varepsilon_i)$, for some sequence $\varepsilon_i \to 0$ as $i \to \infty$. We may then unwrap the collapse, (i.e. resolve the degeneration), by considering the universal covers $\widetilde{B}'_{x_i}(1-\varepsilon_i)$. This sequence does not collapse anywhere. Thus, a subsequence of $(\widetilde{B}'_{x_i}(1-\varepsilon_i), g_i, x_i)$ converges to a limit $L^{2,2}$ Riemannian manifold $(\widetilde{B}'_x(1), g', x)$; see [An2, §3] for further details here.

The L^2 curvature radius is essentially invariant under coverings. However, in the process of passing to the universal cover, we lose the property that

$$\int |\xi_i'|^2 dV_i' = \rho(x_i) \cdot \int |\xi_i|^2 dV_i \to 0, \tag{4.2}$$

see (3.12). In passing to the universal cover, the apriori L^2 bound (2.11) on ξ is lost. The property (4.2) was crucial in the case of convergence. The fact that $\xi' \to 0$ in L^2 as in Proposition 3.1 implies that the limit is a solution of the static vacuum Einstein equations. Without this property, the limit does not satisfy any particular equation.

Recall that, under the assumption (3.43),

$$\nu_i \geq \nu_o \cdot \rho_i, \tag{4.3}$$

the previous results of §3 hold in U_{δ_o} for an *arbitrary* sequence of Yamabe metrics $\{g_i\}$ with a bound on the L^2 norm of z^T. Even though the sequence $\{g_i\}$ satisfies no particular equation (PDE), (besides being a Yamabe metric), the blow-up limits modeling degenerations of $\{g_i\}$ do satisfy a strong PDE, namely the static vacuum equations. It seems quite unreasonable to believe this without some assumption like (4.3). Note that the integral bound $\int |\xi|^2 \leq \frac{s^2}{3}$ from (2.11), which gives rise to (4.2) in blow-ups, becomes less and less meaningful in regions where (M, g_i) is more and more collapsed, since the bound may then come primarily from the volume collapse and not reflect any particular behavior of ξ itself.

This difficulty can be overcome if one had apriori L^∞ control on ξ in place of an L^2 bound, (since such bounds are invariant under coverings), or if the sequence of Yamabe metrics satisfies some other (stronger) P.D.E. This latter will be the path taken in [AnII].

§4.2. Next we make some remarks on the hypothesis (iii) of Theorem A. Suppose $\{g_i\}$ is a degenerating sequence of unit volume Yamabe metrics on M, so that $||z_{g_i}||_{L^2(M)} \to \infty$. It is natural to ask if then (iii) is satisfied. In general, that is for arbitrary sequences, the answer is no, see §6 for further discussion.

Suppose for instance $\{g_i\}$ is a unit volume maximizing sequence for the functional $v^{2/3} \cdot s$, so that

$$s_{g_i} \to \sigma(M). \tag{4.4}$$

We will see in §7, c.f. Theorem 7.2 and Lemma 7.4, that any such sequence can be perturbed slightly if necessary, for instance in the $T^{2,2}$ topology, so that the sequence is still maximizing, and so that the gradient $-z^T = \nabla(v^{2/3} \cdot s|_{\mathcal{C}})$ goes to 0 in the natural dual topology, i.e. at g_i,

$$||z^T||_{T^{-2,2}(T\mathcal{C})} = \sup\left\{ \left| \int_M \langle z^T, \alpha \rangle dV \right| : \alpha \in T\mathcal{C} \text{ and } ||\alpha||_{T^{2,2}} \leq 1 \right\} \to 0, \tag{4.5}$$

as $i \to \infty$. In other words, the sequence $\{g_i\}$ is Palais-Smale for the functional $\mathcal{S}|_{\mathcal{C}}$ w.r.t. the $T^{2,2}$ metric on $\mathbb{M}$. In particular, there are many such Palais-Smale sequences.

Consider now sequences $\{g_i\} \in \mathcal{C}_1$ which satisfy the stronger condition that

$$||z^T||_{T^{-2,2}(T\mathbb{M})} \to 0; \tag{4.6}$$

in other words, the condition (4.5), but where α is no longer constrained to lie in $T\mathcal{C}$. Such sequences will be called *strongly Palais-Smale*. It follows easily from the definition that then also

$$\text{tr } z^T \to 0 \qquad \text{in } T^{-2,2}. \tag{4.7}$$

Under these circumstances, one has the following result.

Proposition 4.1. *Let $\{g_i\}$ be a sequence of unit volume Yamabe metrics satisfying the estimate (4.7), and suppose $\sigma(M) < 0$, or just $s_{g_i} \leq s_o < 0$. Then for the associated functions $u = u_i$, one has*

$$||\tfrac{u-\lambda}{\lambda}||_{L^2} \to 0, \quad \text{as } i \to \infty, \tag{4.8}$$

where the L^2 norm is w.r.t. the metric $g = g_i$.

Proof. Consider the trace equation (2.13) for u, that is, using (2.33),

$$2\Delta\,(u-\lambda) + s\,(u-\lambda) = \lambda\,\mathrm{tr}\; z^T. \tag{4.9}$$

Set $v = \frac{u-\lambda}{\lambda}$ and let $\phi = \phi_i$ be the solution on $(M,g) = (M,g_i)$ to the equation

$$\Delta\phi - \phi = \frac{v}{||v||_{L^2}}. \tag{4.10}$$

Since 1 is not in the spectrum of $\Delta, (\Delta$ is a non-positive operator), this equation has a unique solution. To estimate the $T^{2,2}$ norm of ϕ, pair equation (4.10) with ϕ and integrate by parts to obtain

$$\int |d\phi|^2 + \int \phi^2 = -\int \phi\frac{v}{||v||_{L^2}} \leq \Big(\int \phi^2\Big)^{1/2}. \tag{4.11}$$

Further, squaring both sides of (4.10) gives

$$\int (\Delta\phi)^2 + 2|d\phi|^2 + \phi^2 = 1. \tag{4.12}$$

Thus, the $T^{2,2}$ norm of ϕ is bounded by 1.

It follows that ϕ is an admissible test function in (4.7), so that from (4.9), one obtains

$$\int 2\phi\Delta v + \phi s v \to 0.$$

Integrating by parts gives

$$\int 2v\left[\frac{v}{||v||_{L^2}} + \phi\right] + \phi s v \to 0,$$

or,

$$2||v||_{L^2} + \int (2+s)\phi v \to 0. \tag{4.13}$$

But from (4.10)

$$||v||_{L^2}\Big(\int |d\phi|^2 + \int \phi^2\Big) = -\int \phi v,$$

so that,

$$2||v||_{L^2} - (2+s)||v||_{L^2}||\phi||_{L^{1,2}} \to 0.$$

This implies

$$||v||_{L^2}(2-(2+s)||\phi||_{L^{1,2}}) \to 0. \tag{4.14}$$

Thus either $||v||_{L^2} \to 0$, as required, or

$$||\phi||_{L^{1,2}} \to \frac{2}{2+s}. \tag{4.15}$$

This of course implies $s > -2$, and since $\sigma(M) < 0$, $\frac{2}{2+s} > \frac{2}{2+\sigma(M)} > 1$, (for all i), which with (4.15) contradicts (4.11). Hence the result follows.

■

Note that this argument makes strong use of the $T^{2,2}$ norm; it is not (likely to be) valid w.r.t. the $L^{2,2}$ norm. It is quite easy to see that there are counterexamples to Proposition 4.1 when $\sigma(M) > 0$, c.f. §6.4. If $\sigma(M) = 0$, or more precisely $\{g_i\}$ is a sequence of Yamabe metrics with $s_{g_i} \to 0$, then this result is borderline. Namely, the same argument as above gives either the conclusion (4.8), or (from (4.15), $||d\phi||_{L^2} \to 0$ and $||\phi||_{L^2} \to 1$, with $\int \phi = 0$. The latter case is of course impossible if λ_1, the lowest non-zero eigenvalue of Δ, is bounded away from 0, but without some assumption of this kind, it is not clear if (4.8) follows from (4.7).

Now in case (4.8) holds, it is easy to see that condition (iii) of Theorem A follows from the non-collapse assumption (0.18) or (3.43) and the general degeneration assumption (0.8), (in place of (0.20)). In fact, (compare with Lemma 3.11 and (3.47)), we have

Proposition 4.2. *Let (M, g_i) be a sequence of unit volume Yamabe metrics satisfying the non-collapse assumption (3.43) together with (4.8).*

If $\{g_i\}$ degenerates on M, i.e. if there are points $p_i \in (M, g_i)$ such that $\rho(p_i) \to 0$, then there are points $x_i \in (M, g_i)$ such that

$$|u_i(x_i)/T_i - 1| \to 0, \quad \text{as } i \to \infty, \tag{4.16}$$

and

$$\rho(x_i) \to 0, \quad \text{as } i \to \infty, \tag{4.17}$$

where $T_i = \sup u_i$.

Proof. Note that given (4.8), we do not assume $\sigma(M) < 0$. Note also that $u/T = (\frac{u}{\lambda})/(\frac{T}{\lambda})$.

First, suppose $T/\lambda = T_i/\lambda_i \to \infty$. We claim that any sequence $\{x_i\}$ satisfying (4.16) must then satisfy (4.17). For if not, then for some $\rho_o > 0$ and for some sequence x_i as above,

$$\rho(x_i) \geq \rho_o. \tag{4.18}$$

From (4.18) and the non-collapse assumption, it follows that $B = B_{x_i}(\rho_o)$ has $L^{2,2}$ bounded geometry. Thus, by Theorem 2.10, z^T is uniformly bounded in L^2. Consider the trace equation (4.9),

$$2\Delta\big(\frac{u-\lambda}{\lambda}\big) + s\big(\frac{u-\lambda}{\lambda}\big) = \text{tr } z^T. \tag{4.19}$$

The right side of (4.19) is uniformly bounded in L^2, so that L^2 elliptic estimates [GT, Thm.8.8] imply a bound

$$||v||_{L^{2,2}(D)} \leq C[||v||_{L^2(B)} + 1], \tag{4.20}$$

for $D \subset\subset B$ and $v = \frac{u-\lambda}{\lambda}$. The right side of (4.20) is bounded by (4.8) so that Sobolev embedding on B implies that $\sup u/\lambda = T/\lambda$ is bounded. Thus, under the assumptions above, (4.18) cannot hold.

The same argument as above holds if only $\limsup T_i/\lambda_i > 1$. Namely, this assumption together with (4.20) and Sobolev embedding then implies that, in some subsequence, $u/\lambda = u_i/\lambda_i \geq 1 + \varepsilon$,

for some $\varepsilon > 0$, on a ball $D \subset B$ whose volume is uniformly bounded below, if (4.18) held. This is impossible since $u_i/\lambda_i \to 1$ in L^2 by (4.8). Thus, in the following, we may assume

$$\lim T_i/\lambda_i = 1. \tag{4.21}$$

For a given $t < 1$, let $U_i = U_i(t) = \{x \in (M, g_i) : u_i(x)/\lambda_i \geq t\}$. Suppose that there were $\rho_o > 0$ such that

$$\rho(x) > \rho_o, \qquad \forall x \in U_i. \tag{4.22}$$

Since $u_i/\lambda_i \to 1$ in L^2, it follows that $\mathrm{vol}(M \setminus U_i) \to 0$. Then, arguing as above on (4.19), one also has $u_i/\lambda_i \to 1$ pointwise on compact subsets of each $B = B_{x_i}(\rho_o)$, $x_i \in U_i$. Observe that these balls must cover M. For if not, then there exist balls $B_{q_i}(\rho_o)$ disjoint from U_i. On the other hand, by the non-collapse assumption (3.43), such balls have a definite volume, and hence $B_{q_i}(\rho_o)$ must have intersected U_i. It follows (from (1.27)), that

$$\rho(x) \geq \rho_o/2, \quad \forall x \in M,$$

which contradicts the assumption $\rho(p_i) \to 0$, for some $p_i \in (M, g_i)$. This means that (4.22) does not hold, so that for all $t < 1$, there exist $x_i(t) \in U_i(t)$ such that $\rho(x_i(t)) \to 0$ as $i \to \infty$. Since t is arbitrary, one may choose a sequence $t_i \to 1$ suitably slowly as $i \to \infty$ to give (4.16) and (4.17).

■

We note that Proposition 4.2 does not require λ to be bounded away from 0, or any bound on the L^2 norm of z^T.

Also, note that if $T_i/\lambda_i \to \infty$, then the proof above does not require the hypothesis (4.8), at least when $\sigma(M) < 0$ Hence, in this case, the main hypothesis (3.60) of Theorem 3.10 is automatically satisfied if $\{g_i\}$ degenerates somewhere on M.

Although the estimate (4.8) follows from (4.7) when $\sigma(M) < 0$, it is not clear how to construct, on general manifolds M, sequences of Yamabe metrics satisfying (4.6) or (4.7), even though the Palais-Smale condition (4.6) is easily realized. In §6.5, we construct examples of sequences satisfying (4.6); in fact these sequences satisfy much stronger conditions, c.f. also the discussion in §7.

Remark 4.3. We will not focus here on any applications of Theorem 3.10, but mention one kind of potential application. Suppose $\{g_i\}$ is a sequence of unit volume Yamabe metrics, for instance a maximizing sequence, satisfying the conditions of Theorem A(II). Suppose further that it can be proved on other (geometric or topological) grounds that there are no non-trivial static vacuum solutions arising from blow-ups $g_i' = \rho(x_i)^{-2} \cdot g_i$. (As an example of this, we mention that it can be shown that there are no blow-up limits (N, g_o) which are non-trivial static vacuum solutions with either smooth or compact boundary ∂N, (c.f. §5), provided the sequence (M, g_i) satisfies a uniform Sobolev inequality, as for instance in (3.97), and M is irreducible, i.e. every 2-sphere in M bounds a 3-ball).

Under these conditions, Theorem A immediately implies that one has the following strong estimate

$$\rho(x) \geq c\Big(\frac{|u(x)|}{\sup |u|}\Big), \tag{4.23}$$

whenever $u(x) \neq 0$. If further (4.8) holds, then Proposition 4.2 implies that (4.23) can be strengthened to a bound $\rho(x) \geq c > 0$ everywhere on (M, g_i), i.e. there is no degeneration.

5. Completeness of the Blow-up Limits.

Theorem 3.10 proves under certain conditions the existence of non-flat blow-up limits for the sequence $\{g_i\}$ which model the degeneration of $\{g_i\}$ in a neighborhood of the 0-level of u. In particular, these blow-ups (D, g') are non-trivial solutions to the static vacuum Einstein equations. However, such solutions have only been defined locally. We now address the issue of the completeness of the solutions, together with some aspects of the behavior of the potential function $\bar{u}$ at the boundary and the renormalizations of $\{u_i\}$ used in the construction of $\bar{u}$.

We first discuss the issue of completeness. Recall from Theorem 3.10 that the metric g' is a limit of rescalings of the metrics g_i, i.e. $g' = \lim g'_i$ based at y_i, where

$$g'_i = \rho(y_i)^{-2} \cdot g_i, \tag{5.1}$$

Further, u_i is renormalized to $\bar{u}_i$ satisfying

$$\bar{u}_i(y_i) = 1, \tag{5.2}$$

c.f. (3.69). For a fixed $\mu > 0$ small, let

$$N_i(\mu) = \{x_i \in (M, g'_i) : \rho'(x_i) \geq \mu \cdot \rho'(y_i) = \mu\}, \tag{5.3}$$

where ρ' is the L^2 curvature radius w.r.t. g'_i. Let $N_i(\mu, y_i)$ be the component of $N_i(\mu)$ containing the base point y_i. By Theorem 1.5 and the non-collapse assumption (3.43), as previously discussed in §3.4, the pointed sequence of manifolds $\{(N_i(\mu, y_i), g'_i, y_i)\}$ has a subsequence converging, weakly in $L^{2,2}$ on compact subsets, to a non-flat connected limit domain $(N(\mu, y'), g', y')$. The convergence is in the strong $L^{2,2}$ topology away from the locus $\{|\bar{u}_i| \leq \varepsilon\}$, ε small, and away from the metric boundary of $N_i(\mu, y_i)$, i.e. at any given small distance away from these sets. This limit clearly extends the limit (D, g', y') given by Theorem 3.10. The $L^{2,2}$ limit metric g' on $N(\mu, y')$ induces a natural embedding of any domain K with smooth and compact closure in $N(\mu, y')$ into M, c.f. §1.4.

Now choose a sequence $\mu = \mu_j \to 0$ and consider the double sequence $N_i(\mu_j, y_i)$. There is a diagonal subsequence $\{(i, j_i)\}$ of the double sequence $\{(i, j)\}$, with $j_i \to 0$ sufficiently slowly as $i \to \infty$, so that $\{N_i(\mu_{j_i}, y_i), g'_i, y_i\}$ converges, uniformly on compact subsets in the weak $L^{2,2}$ topology, to a limit (N, g', y'). Although the limit may apriori have more than one component, (the connected domains $N_i(\mu_{j_i}, y_i)$ may pinch off into several limit components), we assume that (N, g', y') is the component of the limit containing the base point y'. The limit N is contained in the union of the inclusions $N(\mu_k, y') \subset N(\mu_{k+1}, y')$, for a sequence $\mu_k \to 0$, and is an open Riemannian manifold with $L^{2,2}$ metric g'. Any smooth open domain K, with smooth boundary, and properly contained in the connected limit N, is embedded as an open domain in M via the metric g'. It is clear from thw work in §3 that the limit functions $\bar{u}_i$ converge, uniformly on compact subsets, to a non-constant limit potential function $\bar{u}$ on N.

We call $(N, g', y', \bar{u})$ the *maximal solution* of the static vacuum equations associated to the buffered sequence $\{y_i\}$. Observe that this maximal solution determined by the (convergent) sequence $\{y_i\}$ is unique, (up to isometry), since any maximal limit must contain the initial domain (D, g') and static vacuum solutions have unique extensions. Hence each domain $N_i(\mu, y_i)$ converges in the pointed Hausdorff topology, (c.f. [G, Ch.5A]), to its limit domain in N.

Let ∂N denote the metric boundary of (N, g'), i.e. the set of (ideal) limits of g'-Cauchy sequences, non-convergent in N. The boundary ∂N might be empty, for instance perhaps if N is the complete (isometrically doubled, c.f. Remark 5.3(ii)) Schwarzschild metric (0.17), but in 'most' cases one expects ∂N to be non-empty. In fact one expects ∂N to often be singular in the sense that the Riemannian metric g' does not extend to ∂N. Of course the completion $\bar{N} = N \cup \partial N$ is a complete metric space w.r.t. the length metric induced by g'.

In case $\partial N \neq \emptyset$, we need to describe ∂N also as a set of (ideal) limits of points of $\{(M, g_i')\}$. For $p \in \bar{N} = N \cup \partial N$, and $p_i \in M$, define

$$p_i \to p, \tag{5.4}$$

if, for all $\delta > 0$, there exists $q = q_\delta \in N$, $\mu = \mu_\delta > 0$ and a sequence $q_i \in (N_i(\mu, y_i), g_i')$ with $\mathrm{dist}_{g_i'}(p_i, q_i) \leq \delta$, such that $q_i \to q$, in the $L^{2,2}$ convergence of $N_i(\mu, y_i) \to N(\mu, y')$ discussed above. It follows from the convergence and the definition of metric completion that $\mathrm{dist}_{g'}(p, q_\delta) \leq \delta$. In particular, since δ is arbitrarily small, p in (5.4) is uniquely determined, in that there do not exist two distinct points p and p' to which the sequence p_i converges.

A sequence of subsets $Z_i \subset (M, g_i')$ is said to converge in the *Hausdorff topology based at* y_i to a set $Z \subset \bar{N} = N \cup \partial N$ if, for each $z \in Z$, there exist $z_i \in Z_i$ such that $z_i \to z$ in the sense of (5.4), and conversely any bounded sequence of points $z_i \in Z_i$, (i.e. $\mathrm{dist}_{g_i'}(z_i, y_i) \leq C$ for some C), has a subsequence converging to a limit $z \in Z$ in the sense of (5.4). This definition extends the notion of Hausdorff convergence of the domains $N(\mu, y_i)$ above to the boundary ∂N.

Observe that ∂N is formed from base points of higher order curvature concentration than $\{y_i\}$. Thus, $p \in \partial N$ if and only if there is a sequence $p_i \in (M, g_i)$, with $p_i \to p$ in the sense of (5.4) with $\rho'(p_i) \to 0$, or equivalently, $\rho(p_i) << \rho(y_i)$.

The next result shows that, in a certain sense, N is defined and complete at least up to the Hausdorff limits of the ε-levels L^ε of $\bar{u}_i$.

Theorem 5.1 **(Completeness).** *Let $(N, g', y', \bar{u})$ be the maximal solution of the static vacuum Einstein equations associated to the (ρ, c) buffered sequence $\{y_i\}$ in (5.1); (hence the non-collapse assumption (3.43) is assumed). Let $\bar{U}_i^\varepsilon$ be the component of $\{x_i \in (M, g_i') : \bar{u}(x_i) > \varepsilon\}$ containing the base point y_i, for $\bar{u}_i$ normalized as in (5.2).*

Let $N^o \subset N$ be the maximal domain on which the potential function $\bar{u} > 0$. Then there is a sequence $\varepsilon_i \to 0$ such that N^o is contained in the y_i-based Hausdorff limit of the domains $\bar{U}_i^{\varepsilon_i} \subset (M, g_i)$. Further the metric boundary $\partial N^o \equiv \bar{L}^o$ of N^o w.r.t. g' is contained in the y_i-based Hausdorff limit of the ε_i-levels $\bar{L}_i^{\varepsilon_i} \equiv \partial \bar{U}_i^{\varepsilon_i}$ of $\bar{u}_i$. The union $\bar{N}^o \equiv N^o \cup \partial N^o \subset \bar{N}$ is complete w.r.t. the metric $\bar{g}$ but may be singular at the boundary.

Let $t(x) = \mathrm{dist}_{g'}(x, \partial N^o)$, for $x \in N^o$. Then, as in (3.16), there is an absolute constant $K < \infty$ such that

$$|z|(x) \leq \frac{K}{t^2(x)}, \qquad \bar{u}^{-1}|\nabla \bar{u}|(x) \leq \frac{K}{t(x)} \tag{5.5}$$

Proof. In the notation above, consider any sequence of points $\{z_i\} \in \bar{U}_i^\varepsilon$ with $\mathrm{dist}_{g_i'}(z_i, y_i) \leq R$, for some arbitrary (large) constant $R < \infty$. Suppose further that

$$\mathrm{dist}_{g_i'}(z_i, \bar{L}_i^\varepsilon) \geq \delta, \tag{5.6}$$

for an arbitrary (small) constant $\delta > 0$, where $\bar{L}_i^\varepsilon = \partial \bar{U}_i^\varepsilon$.

It follows from Theorem 3.3 that there is a constant $a = a(\varepsilon) > 0$ such that

$$\rho'(z_i) \geq a \cdot \delta, \tag{5.7}$$

Thus, the curvature of $\{g'_i\}$ does not blow-up (in L^2) within bounded distance to $\{y_i\}$, provided one stays a fixed distance away from the level set $\bar{L}_i^{\varepsilon}$ in $\bar{U}_i^{\varepsilon}$. Note also that the assumption (3.43) prevents collapse in these balls $B'_{y_i}(\rho'(z_i))$.

Let $N'_i = N'_i(\varepsilon, \delta)$ be the connected component of the set $\{z_i \in (M, g'_i) : z_i \in \bar{U}_i^{\varepsilon}, \mathrm{dist}(z_i, \bar{L}_i^{\varepsilon}) \geq \delta\}$ containing the base point y_i. Using Theorem 1.5 and Theorem 3.4 in the usual way, the pointed Riemannian manifolds $\{(N'_i, g'_i, y_i)\}$ (sub)-converge in the strong $L^{2,2}$ topology based at $\{y_i\}$ to a limit smooth metric g', based at y', defined on an open domain $N'(\varepsilon, \delta)$; the convergence is uniform on compact subsets. The functions $\bar{u}_i$ converge to a limit harmonic function $\bar{u}$ on $N'(\varepsilon,\ \delta), \bar{u} \geq \varepsilon$, so that the pair $(g', \bar{u})$ is a non-flat solution of the static vacuum Einstein equations on $N'(\varepsilon, \delta)$. Of course $y' \in N'(\varepsilon, \delta)$ for any ε, δ small and $\bar{u}(y') = 1$. Given sequences $\varepsilon_j \to 0$ and $\delta_j \to 0$, there exist suitable diagonal subsequences $j = j_i$ of the double sequence $N'_i(\varepsilon_j, \delta_j)$, (with ε_{j_i} and $\delta_{j_i} \to 0$ sufficiently slowly as $i \to \infty$), which converge, uniformly on compact subsets, to a limit. As above in the construction of N, we consider only the component (N^o, g') of the limit containing y'. The connected limit (N^o, g', y') is an is an open Riemannian manifold, and is contained in the union of the domains $N'(\varepsilon_k, \delta_k) \subset N'(\varepsilon_{k+1}, \delta_{k+1})$ for some sequence $k \to 0$. The limit potential function $\bar{u}$ on N^o satisfies $\bar{u} \geq 0$ on N^o, and $\bar{u}(y') = 1$. Thus, $\bar{u} > 0$ by the minimum principle for harmonic functions. As discussed in §1.3, (N^o, g') is hence a smooth solution of the static vacuum equations.

The construction of N and N^o implies that

$$(N^o, g') \subset (N, g'), \tag{5.8}$$

(for any choice of subsequence above) and N^o is unique. It is possible, although not necessary, that $N^o = N$. It follows immediately from the definition preceding Theorem 5.1, (c.f. also the argument below), that (N^o, g') is contained in the y_i-based Hausdorff limit of the domains $(\bar{U}^{\varepsilon_i}, g'_i)$, where $\varepsilon_i = \varepsilon_{j_i}$ is defined above.

Let ∂N^o be the metric boundary of N^o w.r.t. g' and let $\bar{N}^o = N^o \cup \partial N^o$ be the metric completion. Of course $\partial N^o \subset \bar{N}$.

We claim that ∂N^o is contained in the y_i based Hausdorff limit of the levels $\bar{L}^{\varepsilon_i},\ \varepsilon_i = \varepsilon_{j_i}$. To see this, suppose $p \in \partial N^o$, so that $p = \lim p_j$, for $\{p_j\}$ a Cauchy sequence in N^o. Thus, $p_j \in N'(\varepsilon_j, \delta_j)$, for some $\varepsilon_j, \delta_j > 0$. But $N'(\varepsilon_j, \delta_j) = \lim_{i\to\infty} N'_i(\varepsilon_j, \delta_j)$ and hence for any j, there exist sequences $q_{i,j} \in N'_i(\varepsilon_j, \delta_j)$ with $q_{i,j} \to p_j$ as $i \to \infty$, as required.

Suppose first $\bar{N}^o \subset N$, (so that in particular $\partial N^o \cap \partial N = \emptyset$). Then the metric g' extends as an $L^{2,2}$ metric past $\bar{N}^o$ and the convergence $g'_i \to g'$ is in the weak $L^{2,2}$ topology in an open region R satisfying $\bar{N}^o \subset R \subset N(\mu, y') \subset N$, for some $\mu > 0$. (More precisely, μ may depend also on the distance to the base point y' if ∂N^o is non-compact). Since the functions $\bar{u}_i$ also converge in $L^{2,2}$ in R, (in fact strongly in $L^{2,2}$ by (3.35)), it follows that ∂N^o is identified with the 0-level set of $\bar{u}$, i.e. with the event horizon Σ of (N, g').

In this case, since g' is $L^{2,2}$ in a neighborhood of the event horizon Σ, as noted in §1.3, (c.f. the discussion following Theorem 1.1), elliptic regularity arguments show that in fact g' is C^∞ smooth across Σ and Σ is a collection of smooth totally geodesic surfaces forming the (smooth) topological

boundary of N^o. Regardless of this fact, the weak maximum principle for $L^{2,2}$ harmonic functions, c.f. [GT, Thm. 8.1], implies that $\bar{u} < 0$ in $N \setminus \bar{N}^o$.

More generally, since the discussion above is local, it holds for the parts of ∂N^o which are contained in N, i.e. those points of ∂N^o which admit an open neighborhood contained in N. Now both ∂N and ∂N^o are closed, so the remaining points of ∂N^o are those contained in ∂N. Thus, the domain N^o is the maximal domain in N on which $\bar{u} > 0$. The set $Z = \partial N^o \cap \partial N$ need not be empty, and one would frequently expect $\partial N^o = \partial N$.

To conclude, the estimates (5.5) follow immediately from Theorem 3.2(II), (applied to any smooth subdomain in N^o), and the fact that $\bar{u} > 0$ on N^o, c.f. also (A.22) and the discussion following it at the end of the Appendix.

■

Note that the limit function satisfies $\bar{u}(y') = 1$, since $\bar{u}(y_i) = 1$ and, since $\rho'(y_i) = 1$, the convergence to the limit is controlled near y_i. In particular, ∂N does not intersect $B_{y'}(1)$ and if ∂N^o intersects $B_{y'}(1)$, it does so smoothly, as noted above.

However, it is apriori possible that all levels $\bar{L}_i^{\varepsilon}$ approach for instance the level $\bar{L}_i^1$ as $i \to \infty$ away from $B'_{y_i}(1)$. Thus, although the domains $N'_i(\varepsilon, \delta)$ contain $N'_i(1, \delta)$ for $\varepsilon < 1$, they may apriori give rise to the same limit domains, outside some region containing $B_{y'}(1)$. In other words, the functions $\bar{u}_i$ might descend from the value 1 (for example) to a value arbitrarily near 0 in arbitrarily short g'_i distances as $i \to \infty$. Of course, this can only occur at points where $\rho' \to 0$, i.e. at ∂N. For the same reasons, we do not assert that $\bar{u}(x_j)$ approaches 0 whenever $x_j \in N^o$ approaches $\partial N^o \equiv \bar{L}^o$.

Thus, $\bar{L}^o$ cannot necessarily be identified as the 0-level set of the potential $\bar{u}$, i.e. with the event horizon. Further remarks on the structure of the maximal limits (N^o, g') and (N, g') follow below.

Remark 5.2. We conjecture that for the blow-up limits (N, g', y') given by Theorem 5.1, one has

$$\partial N^o = \{\bar{u} = 0\} = \Sigma, \tag{5.9}$$

so that ∂N^o is identified with the event horizon of the static vacuum solution. This amounts to proving that $\bar{u}(x_k) \to 0$ whenever x_k converges to a point in ∂N^o.

Remark 5.3. (i). In general relativity, it is usually assumed that a static vacuum solution is complete up to the event horizon $\Sigma = \{\bar{u} = 0\}$, and in particular that $\bar{u}(x) \to 0$ as $x \to \Sigma$. Such static vacuum solutions are the most natural physically. As noted in §1.3, the static vacuum equations are formally degenerate at the locus Σ, but are formally non-degenerate away from Σ. By Theorem 3.2(I), there are no non-trivial complete solutions with empty Σ.

There are however examples of static vacuum solutions which are not complete (or defined) up to Σ, c.f. [An4, §2]. For instance, let ν be a bounded harmonic function on $\mathbb{R}^3 \setminus B(1)$, with axially symmetric, but discontinuous or non-smooth boundary values on $\partial B(1)$. Then ν is axially symmetric and hence defines a Weyl solution as in (1.17), which does not extend everywhere past $B(1)$. For suitable boundary values, the solution will not extend anywhere into $B(1)$, and hence one has a solution complete away from $B(1)$, but not defined up to the event horizon, (where $\nu = -\infty$).

Note that such examples may have $\max \nu$, and so $\max u$, occurring on $\partial B(1)$, while $\min \nu$ occurs at infinity. In this case, such solutions formally have 'negative mass', c.f. §1.3.

Of course, from Remark 5.2, we conjecture that such solutions cannot arise as blow-up limits of unit volume Yamabe metrics.

(In the opposite direction, there are also solutions which are defined and smooth up to Σ, but which are not complete away from Σ, for example the A2 or B2 solutions, c.f. [EK, §2-3.6]).

(ii). While there are many static vacuum solutions which are singular at Σ, but smooth and complete away from Σ, the Schwarzschild metric (0.17) is of course smooth up to and at Σ. Here Σ is given by a round, totally geodesic 2-sphere S^2. Thus, the Schwarzschild metric g_s may be isometrically doubled across Σ giving a complete, smooth metric on $S^2 \times \mathbb{R}$, asymptotically flat at both ends. This is clearly the maximal (abstract) smooth extension of g_s, call it $\tilde{N}$. However, it is not necessarily the case that $\tilde{N}$ agrees with the maximal extension N defined preceding Theorem 5.1. Apriori, it is possible that the curvature of g_i' blows up near the event horizon Σ, (where one loses strong convergence), even though the limit in this particular case is smooth across Σ.

Basically because of this, we will usually consider only the behavior of limits up to the event horizon, (in case this is defined), i.e. within the domain $N^o \subset N$; c.f. however §6.5.

(iii). Although stated only for buffered sequences $\{y_i\}$ and the associated limit, it is clear from the discussion preceding Theorem 5.1 and its proof that Theorem 5.1 remains valid for arbitrary maximal limits (N, g', x), $g' = \lim g_i'$, $g_i' = \rho(x_i)^{-2} \cdot g_i$, for which $\rho(x_i) \to 0$ and say $(u_i/T_i)(x_i) \geq u_o$, for some arbitrary constant $u_o > 0$. Of course in this case, the limits may be flat solutions to the static vacuum equations.

The following result, valid for general static vacuum solutions, shows that the estimate (5.5) can be improved in certain regions, c.f. also (A.27).

Lemma 5.4. *Suppose (N, g, u) is an (arbitrary) static vacuum solution, $u > 0$ on N, and u is bounded above. Let $\{x_j\}$ be a maximizing sequence for u in N, i.e. $u(x_j) \to \sup u < \infty$, and $t(x) = \mathrm{dist}(x, \partial N)$, where ∂N is the metric boundary of N. Then*

$$|z|(x_j) \leq \frac{\mu_j}{t^2(x_j)}, \qquad |\nabla log u|(x_j) \leq \frac{\mu_j}{t(x_j)}, \tag{5.10}$$

where $\mu_j = \mu_j(x_j) \to 0$ as $j \to \infty$.

Proof. Consider the natural rescalings of (N, g) based at $\{x_j\}$, i.e. the sequence of metrics (N, g_j, x_j), where $g_j = t(x_j)^{-2} \cdot g_o$. Let $t_j = t/t(x_j)$ be the distance to ∂N w.r.t. the metric g_j, so that $t_j(x_j) = 1$. Now of course the curvature of $\{g_j\}$ will blow up at any fixed base point $x_o \in N$, (with $z(x_o) \neq 0$), if $t(x_j) \to \infty$, and hence there is no limit in this region. Similarly, for example the curvature of g_j may blow up when $t(x_j) \to 0$.

However, given any $\delta > 0$, the curvature, (w.r.t. g_j), of the domain $N_\delta(j) = \{p_j \in (N, g_j) : t_j(p_j) \geq \delta\}$, containing x_j remains uniformly bounded by the (scale-invariant) estimate (3.16). If the sequence $\{(N_\delta(j), g_j, x_j)\}$ is non-collapsing at x_j, it follows from the (local) Cheeger-Gromov theory, (or Theorem 1.5), that a subsequence of $\{B_{x_j}(\frac{1}{2}), g_j, x_j)\}$ converges to a limit $(B_\infty, g_\infty, x_\infty)$. By Lemma A.2, the convergence is smooth and uniform on compact subsets, and the limit is again a smooth solution of the static vacuum equations. If the sequence instead collapses at x_j, then the collapse may be unwrapped by passing to the universal cover $(\widetilde{B}_{x_j}(\frac{1}{2}), g_j)$; this sequence no longer collapses, and hence sub-converges to a limit as above. (We refer to the Appendix - Lemma A.2 and Corollary A.3 - for more details of this type of standard argument).

On the other hand, it is clear that $u(x_\infty) = \sup u$, and so by the maximum principle, the limit harmonic function u is constant on B_∞ and g_∞ is hence flat. Since the convergence of $N_\delta(j)$ to the limit is smooth, the curvature of g_j is almost 0, and u is almost constant, in $N_\delta(j)$, away from the

boundary. This implies (5.10) by scale-invariance.

■

Next, we prove that the domain $N^o \subset N$ is large in a natural sense; it particular it is unbounded.

Proposition 5.5. *Let (N, g', y') be a maximal non-flat limit solution as in Theorem 5.1, with domain $N^o \subset N$ on which $\bar{u} > 0$. There is a smooth curve $\gamma : \mathbb{R}^+ \to N^o$, parametrized by arclength, with $\gamma(0) = y'$, and positive constants δ_1, δ_2 such that*

$$\mathrm{dist}(\gamma(s), y') \geq \delta_1 \cdot \mathrm{dist}(\gamma(s), \partial N^o), \tag{5.11}$$

and such that the cone $V = V(d_2) = \{x : \mathrm{dist}\,(x, \gamma(s)) \leq \delta_2 \cdot s\}$ over γ satisfies

$$V \subset N^o, \quad i.e. \quad V \cap \partial N^o = \emptyset. \tag{5.12}$$

In particular, the function $t(x) = \mathrm{dist}_{g'}(x, \partial N^o)$ has linear, and hence unbounded, growth in V.

Proof. This follows from the descent construction of the limit N in Theorem 3.10. Thus, recall that the buffered sequence $y_i = x_i^{k_o}$ in (M, g_i'), $k_o = k_o(i) \to \infty$, as $i \to \infty$, has predecessors $x_i^{j'}$, $j' = j'(i,j) = k_o - j$, for any fixed $j' > 0$. Thus, we have relabeled so that $y_i = x_i^{j'}$, for $j' = 0$. By the construction, these have $\bar{u}$-values at least $(\frac{1}{2})^{j'}$, c.f. (3.73). In particular, these points are in $N_i'(\varepsilon, 1)$, for ε sufficiently small, (depending on j'), c.f. the proof of Theorem 5.1; compare also with Remark 3.15(iv). Further, in the scale g_i' associated to y_i these points have very large curvature radius; in fact $\rho'(x_i^{j'}) \geq (d_1)^{-j'}$, for all $j > 0$, while both $\mathrm{dist}_{g_i'}(x_i^{j'}, y_i)$ and $\mathrm{dist}_{g_i'}(x_i^{j'}, \bar{L}^\varepsilon)$ are also on the order of $(d_1)^{j'}$, (for ε sufficiently small, depending on j', c.f. (3.66)-(3.67)). Note that from (1.27), for any $q \in B_{x_i^{j'}}(\rho'(x_i^{j'}))$, one has $\rho'(q) \geq \mathrm{dist}_{g_i'}(q, \partial B_{x_i^{j'}}(\rho'(x_i^{j'})))$.

Thus, let $\gamma(s)$ be a path joining the limit points $x^{j'} = \lim x_i^{j'}$ in N^o, approximating a minimizing geodesic joining each $x^{j'+1}$ to $x^{j'}$ within $B^{j'} = B_{x_{j'}}(\rho'(x^{j'}))$. One may then define

$$V = \bigcup_{j' \geq 0} B_{x^{j'}}(\rho'(x^{j'})) \subset N^o,$$

and the result follows.

■

The curvature estimate (5.5) on N^o can be improved in the region V in (5.11), since the curvature radius has linear growth in V, and the predecessors $x_i^{j'}$ are not (ρ, c) buffered. Thus, in fact

$$|z|(x) \leq \frac{\kappa}{t^2(x)}, \qquad \bar{u}^{-1}|\nabla \bar{u}|(x) \leq \frac{\kappa}{t(x)} \tag{5.13}$$

for $x \in V$, where $\kappa = \kappa(c)$ may be made small by choosing the buffer constant c sufficiently small. This follows since (5.13) holds in an L^2 sense, from the fact that x is not (ρ, c) buffered, together with elliptic regularity for the static vacuum equations, which gives an L^∞ bound in terms of an L^2 bound.

It seems possible that Proposition 5.5 may not hold for arbitrary 'complete' static vacuum solutions. (For example, consider the positive measure μ formed by placing positive multiples of the Dirac measure at all integer lattice points in $\mathbb{R}^3$, weighted so that the total mass is finite. Let v be the Newtonian potential of μ. Then there may possibly be static vacuum solutions whose potential u resembles the geometry of v).

Remark 5.6. We conclude with some remarks on the issue of the renormalization of $\{u_i\}$ and the relation of $\bar{u}$ with the initial sequence $\{u_i\}$.

Recall that the static vacuum solution constructed in Theorem 3.10 may well live in the region of (M, g_i') where u_i is converging uniformly to 0; the potential function $\bar{u}$ of the static limit is obtained by renormalizing, (possibly infinitely many times), the original sequence $\{u_i\}$. This will occur for instance if u_i behaved as the function $v_i = t^{\delta_i}$, with $\delta_i \to 0$ as $i \to \infty$, and $t(x) = \text{dist}(x, L^o)$. Namely, if x_ε satisfies $v_i(x_\varepsilon) = \varepsilon$ for any fixed $\varepsilon > 0$, and the distance between L^ε and L^o is scaled to size 1, then v_i approaches the constant function ε uniformly near x_ε as $\delta_i \to 0$.

In fact, we point out that there are static vacuum solutions (N, g, u) and points $x_j \in N$ with $t(x_j) = 1$, ($t(x)$ = distance to event horizon), such that

$$u(z_j) \geq t^{\delta_j}(z_j), \tag{5.14}$$

for a given sequence $\delta_j \to 0$, for all points $z_j \in B_{x_j}(2)$. Namely, consider for example the Weyl solution (1.17) with potential function ν determined by the Riesz measure

$$d\mu_\zeta = \frac{1}{1 + |\zeta|}\, dL, \tag{5.15}$$

where dL is Lebesgue measure on the axis A, and ζ is a parameter for A, c.f. (1.18). This solution is complete up to the event horizon, so $\bar{L}^o = \Sigma = \{u = 0\}$; the event horizon corresponds to the full axis A. If x_j diverges to infinity in (N, g, u) at any fixed distance to Σ, then the potential function $u = e^\nu$ satisfies $u(x_j) \to 1 = \sup u$ and has the property (5.14).

Thus, in addition to the discussion in Remark 5.9, this shows that the infinite descent down the u-levels in Theorem 3.10 is necessary.

It is worth noting that in this example, Σ is highly singular, and the Riesz measure $d\mu$ converges to 0 at infinity on the axis $A \sim \Sigma$. In fact, any Weyl solution whose Riesz measure has regions of arbitrarily long length on A, with arbitrarily small, but non-zero, density w.r.t. Lebesgue measure on A will have behavior similar to (5.14).

On the other hand, although one must carry out the inductive construction in Theorem 3.10 arbitrarily many times, i.e. $k_o(i) \to \infty$, as $i \to \infty$, this does not *necessarily* imply an infinite descent down the levels L^k of u. Recall that $u(x_i^k) \geq 2^{-k}$, and it may well happen that that in fact $u(x_i^k) \geq 2^{-1}$, for all $k \leq k_o$, (if c is chosen sufficiently small). This is seen specifically for example in the Schwarzschild metric, as in Example 3.9. Thus, choose initial base points $\{x_i^1\}$ going to infinity in the Schwarzschild metric, so that $u(x_i^1) \to 1$. If one carries out the inductive process of Theorem 3.10, then it is necessary to take $k_o(i) \to \infty$ in order to obtain a (ρ, c) buffered sequence $\{y_i\}$ from the sequence $\{x_i^1\}$. However, one easily sees that $u(x_i^k) \geq 2^{-1}$ for all $k \leq k_o$, so that one has $u(y_i) \geq 2^{-1}$ also, (for a suitable choice of c). In this example, the inductive process in the proof of Theorem 3.10 just recaptures the original Schwarzschild metric, (up to a bounded scale change).

Similar behavior holds for Weyl solutions whose Riesz measure on A has density w.r.t. the Lebesgue measure on A either 0 or uniformly bounded below.

6. Construction of Yamabe Sequences with Singular Limits.

In this section, we discuss in detail some constructions of sequences of Yamabe metrics which illustrate the sharpness of the results in §3. In particular, these constructions exhibit the possibility that blow ups of Yamabe metrics may give rise only to trivial (i.e. flat), or super-trivial solutions

of the static vacuum equations, see (3.15). (Discussions with R. Hamilton and R. Schoen were helpful to me in clarifying some aspects of the construction in Example 1). In §6.3, we analyse the behavior of the splittings (2.6) and (2.10) of z and g respectively on the (singular) limits of these sequences.

On the other hand, in §6.4 and §6.5, we construct examples of sequences of Yamabe metrics which do satisfy all the hypotheses of Theorem A(II)/Theorem 3.10, (so that this result is non-vacuous). The existence and basic properties of these four classes of examples are explained from a somewhat more general perspective in §7.

§6.1. Example 1. Let (M, g_o) be a hyperbolic 3-manifold. Clearly, g_o is a critical point of $v^{-1/3} \cdot \mathcal{S}$ on $\mathbb{M}$, and thus of $v^{2/3} \cdot s$ on $\mathcal{C}$. In fact, g_o is a local maximum of $v^{2/3} \cdot s$ on $\mathcal{C}$, c.f. [Bes, 4.60].

Let N be any closed, oriented 3-manifold with $\sigma(N) > 0$, for instance $N = S^3,\ S^3/\Gamma,\ S^2 \times S^1$, or a connected sum of such manifolds. In the following, we will construct Yamabe metrics on the manifold $N \# M$, which are geometrically close to the original hyperbolic manifold (M, g_o).

Let γ be any smooth metric (not necessarily Yamabe) of positive scalar curvature on N. Such a metric admits a positive Green's function $G_y(x)$, with pole at y, for the conformal Laplace operator $-8\Delta_\gamma + s_\gamma$, c.f. [LP, Thm2.8]. Consider the conformally related metric

$$g = G_y^4(x) \cdot \gamma, \tag{6.1}$$

for any fixed $y \in N$. This metric is a complete, scalar-flat metric on $N \setminus \{y\}$, which is asymptotically flat in the sense that, outside a large compact set, (thus in a small neighborhood of $\{y\}$),

$$g_{ij} = \left(1 + \frac{2m}{r}\right) \delta_{ij} + O(r^{-2}), \tag{6.2}$$

c.f. (1.15). Here m is the mass of the metric satisfying $m \geq 0$, with equality if and only if g is flat, by the positive mass theorem [SY]. From [Sc1], the metric g is flat only when (N, γ) is the canonical constant curvature metric on S^3. Note that the curvature tensor R_g of g satisfies the decay condition

$$|R_g|(x) \leq c \cdot r(x)^{-3}, \tag{6.3}$$

as $r(x) \to \infty$.

Choose any fixed value of ε, with $0 < 4\varepsilon < i_o \equiv \mathrm{inj}_{g_o}(M)$, where inj denotes the injectivity radius. Given any $x \in M$, we may then glue in the metric g above to $B_x(2\varepsilon) \subset (M, g_o)$ as follows. Given a fixed center point $p \in N \setminus \{y\}$, let $B(R)$ denote the geodesic R-ball in $(N \setminus \{y\}, g)$ centered at p. Scale $g|_{B(R)}$ to size ε, i.e. define

$$g_\varepsilon = \left(\tfrac{\varepsilon}{R}\right)^2 \cdot g, \tag{6.4}$$

so that $g_\varepsilon|_{B_x(\varepsilon)}$ is homothetic to $g|_{B_p(R)}$. From (6.2) and (6.3), it follows that in the geodesic annulus $A(\varepsilon, 2\varepsilon)$, the sectional curvature of g_ε is on the order of $|K_\varepsilon| = O(\varepsilon^{-2}R^{-1})$. Hence the curvature is bounded in $A(\varepsilon, 2\varepsilon)$ if $\varepsilon^2 R$ is bounded away from 0, while the metric g_ε is almost flat in this band if $\varepsilon^2 R >> 1$. For simplicity, we assume from now on that $\varepsilon^2 R >> 1$. Note that the unit ball $B_p(1)$ in (N, g) is of radius ε/R in g_ε; for $\varepsilon^2 R > 1$, $\varepsilon/R >> \varepsilon^3$.

In order to bend the metric g_ε on $B(2\varepsilon)$ so it has almost constant scalar curvature $-6 = s_{g_o}$, define

$$\tilde{g}_\varepsilon = \psi^4 \cdot g_\varepsilon, \tag{6.5}$$

where $\psi = 2^{1/2}(1-\tau^2)^{-1/2}$ on $B(2\varepsilon)$ and where τ will be determined below. Note that if g_ε were flat on $B(2\varepsilon)$ and $\tau = t = \mathrm{dist}_{g_\varepsilon}(x, \cdot)$, the metric $\tilde{g}_\varepsilon$ is the hyperbolic metric of scalar curvature -6. Since g_ε is scalar-flat a simple computation using (1.12) shows that

$$\tilde{s}_\varepsilon = -(1-\tau^2)\Delta\tau^2 - 6\tau^2|d\tau|^2. \tag{6.6}$$

We choose τ so that $\Delta\tau^2 = 6$, and τ is close to the distance function $t(z) = \mathrm{dist}_{g_\varepsilon}(z, x)$. Hence (6.6) becomes

$$\tilde{s}_\varepsilon = -6[1 + \tau^2(|d\tau|^2 - 1)]. \tag{6.7}$$

To construct such functions, return to the asymptotically flat manifold $(N \setminus \{y\}, g)$. We claim there is a function α on N, with $\Delta\alpha = 6$ and α asymptotic to r^2. To define α consider the function $r^2 + mr + 2mlnr$; this is defined and smooth outside a compact set K of $(N \setminus \{y\}, g)$, and off K, one computes that $\Delta(r^2 + mr + 2mlnr) = 6 + 4mr^{-3} + 0(r^{-4})$. Let h be a smooth extension of $r^2 + mr + 2mlnr$ to all of $N \setminus \{y\}$; then $\Delta h = f$, where the function f satisfies $|6 - f| = 0(r^{-3})$. We define α as $\alpha = h + \phi$, where $\phi(q) = -\int G(q,z)(6-f)dz$ and G is the (positive) Green's function for Δ on $N \setminus \{y\}$. Since the product $\mathrm{dist}(q,z) \cdot G(q,z)$ is bounded below and above, ϕ is well defined and decays at infinity as $0(r^{-1})$. It follows that the function α satisfies $\Delta\alpha = \Delta(h+\phi) = 6$ on $(N \setminus \{y\}, g)$. By adding a suitable constant, we may assume $\alpha > 0$.

Rescaling to the metric g_ε, we then define $\tau = (\varepsilon/R)\alpha^{1/2}$ on $B_x(2\varepsilon)$. Since the metrics g and g_ε are homothetic, scaling properties imply $\Delta_{g_\varepsilon}\tau^2 = 6$ and $|d\tau|_{g_\varepsilon} = |d\alpha^{1/2}|_g \approx 1$ in $A(\varepsilon, 2\varepsilon)$, while $|d\tau|_{g_\varepsilon}$ is bounded everywhere. Note also that $\tau \to 0$ uniformly in $B_x(2\varepsilon)$ as $\varepsilon \to 0$.

It follows from (6.7) that the metric $\tilde{g}_\varepsilon$ has scalar curvature converging to -6 in $B(2\varepsilon)$ as $\varepsilon \to 0$ and $\varepsilon^2 R \to \infty$. Further, from the remarks following (6.5) the metric $\tilde{g}_\varepsilon$ approaches the hyperbolic metric in $A(\varepsilon, 2\varepsilon)$ in the C^2 topology, as $\varepsilon \to 0$ with $\varepsilon^2 R \to \infty$. Thus, the metric $\tilde{g}_\varepsilon$ may be perturbed a small amount in C^2 in $A(\varepsilon, 2\varepsilon)$ to match with the hyperbolic metric g_o at $S(2\varepsilon)$ to give a smooth metric, again called $\tilde{g}_\varepsilon$ on $N \# M$, with scalar curvature close to -6 everywhere. Note that the conformal factor ψ converges pointwise to the conformal factor ψ_o bending the flat metric to the hyperbolic metric. In particular, on $(B_x(2\varepsilon), g_\varepsilon)$, we have

$$\psi = \psi_\varepsilon \to 2^{1/2}, \qquad \text{as} \quad \varepsilon \to 0, \quad \varepsilon^2 R \to \infty. \tag{6.8}$$

Finally, let $\bar{g}_\varepsilon$ be the Yamabe metric in the conformal class of $\tilde{g}_\varepsilon$ with the same volume. Thus

$$\bar{g}_\varepsilon = w^4 \cdot \tilde{g}_\varepsilon, \tag{6.9}$$

where w satisfies

$$w^5 \bar{s}_\varepsilon = -8\Delta w + \tilde{s}_\varepsilon w. \tag{6.10}$$

Noting that $\min\ w < 1 < \max\ w$ and evaluating (6.10) at points realizing the minimum and maximum of w, gives the estimates

$$\min\ w \geq \inf\left|\frac{\tilde{s}_\varepsilon}{\bar{s}_\varepsilon}\right|, \quad \max\ w \leq \sup\left|\frac{\tilde{s}_\varepsilon}{\bar{s}_\varepsilon}\right|. \tag{6.11}$$

Since $\tilde{s}_\varepsilon$ converges to -6 in the C^o norm as $\varepsilon \to 0$ and $\varepsilon^2 R \to \infty$, each ratio in (6.11) must converge to 1, so that

$$w \to 1 \tag{6.12}$$

in the C^o topology. Thus, the family of Yamabe metrics $\{\bar{g}_\varepsilon\}$ on $N\#M$ have scalar curvatures converging to $-6 = s_{g_o}$, and clearly converge smoothly to the hyperbolic metric on M away from the point x. The manifold N is being crushed to the point x under $\{\bar{g}_\varepsilon\}$, as $\varepsilon \to 0$.

It is easy to see that $\{\bar{g}_\varepsilon\}$ is a degenerating family in the sense of (0.8), that is

$$\int_{N\#M} |z_{\bar{g}_\varepsilon}|^2 \, dV_{\bar{g}_\varepsilon} \to \infty, \quad \text{as } \varepsilon \to 0,$$

provided $(N, \gamma) \neq (S^3, g_{can})$. Namely, in this case, note that the metric g on $N \setminus \{y\}$ satisfying (6.2) has a definite amount of curvature, say

$$0 < \int_{B_p(R)} |z_g|^2 \, dV_g = \kappa < \infty, \tag{6.13}$$

for R large. The scaling properties of curvature then imply that

$$\int_{N\#M} |z_{\tilde{g}_\varepsilon}|^2 \, dV_{\tilde{g}_\varepsilon} \geq \tfrac{1}{2} \int_{B(\varepsilon)} |z_{g_\varepsilon}|^2 \, dV_{g_\varepsilon} = \frac{\kappa}{2} \cdot \frac{R}{\varepsilon} \to \infty, \qquad \text{as } \varepsilon \to 0, \tag{6.14}$$

where the first inequality follows from (6.8) and elliptic regularity applied to the equation defining ψ. (The factor $\frac{1}{2}$ can be replaced by any number < 1). Similarly, the estimate (6.14) then also holds for the Yamabe metrics $\{\bar{g}_\varepsilon\}$, using (6.12) and elliptic regularity for the equation (6.10) defining w.

Observe however that all blow-ups by the L^2 curvature radii of $\{\bar{g}_\varepsilon\}$, as $\varepsilon \to 0$, $\varepsilon^2 R \to \infty$, have either flat limits, or are exactly the limit metric g, given by (6.1) above. It is clear that this metric can only be a super-trivial solution of the static vacuum equations, with potential function u identically zero. For instance, since the metric g is asymptotically flat and smooth everywhere, Theorem 1.1 implies that any non-trivial static vacuum solution would have to be the Schwarzschild metric, which is obviously not isometric to g. Thus all blow-up limits of $\{\bar{g}_\varepsilon\}$ are either trivial, i.e. flat, or are super-trivial. It is easily seen that hypothesis (i) and, using Theorem 2.10, hypothesis (ii) of Theorem A(II) are satisfied. In particular, this shows that the assumption (iii) in Theorem A(II) is necessary.

We make several further remarks on this construction, c.f. also §6.3.

Remark 6.1. (i). It is obvious that this construction can be carried out on any collection of disjoint 4ε balls in (M, g_o). Choosing a sequence $R_i \to \infty$ and $\varepsilon_i \to 0$, so that $\varepsilon_i^2 \cdot R_i \to \infty$, one may construct sequences of Yamabe metrics $\{g_i\}$ on manifolds of the form

$$M_{k_i} = \left(\overset{k_i}{\underset{1}{\#}} N_j \right) \# M,$$

where N_j is any closed oriented manifold with $\sigma(N_j) > 0$ and $\{k_i\}$ either bounded or divergent to ∞. Such sequences satisfy $v^{2/3}s(g_i) \to v^{2/3}s(g_o)$, and may be arranged that their curvature blows up on a progressively denser set in any prescribed domain $(\Omega, g_o) \subset (M, g_o)$.

(ii). One may also glue in metrics of the form (6.1), where the Green's function based at y is replaced by a finite sum of Green's functions based at a finite number of poles $\{y_i\} \in N$. This gives rise to a complete scalar-flat metric with a finite number of asymptotically flat ends E_i, each of

which may be glued as above into a small ball of a hyperbolic manifold M_i. The limit as $\varepsilon \to 0$ is then a finite number of hyperbolic manifolds, glued together at one point.

(iii). It is easy to see that this construction does not require that the gluing be into a hyperbolic manifold (M, g_o). The same procedure is valid on any 3-manifold (M, g) with Yamabe metric g of scalar curvature $s_g < 0$.

Namely, it is a standard result that given any such (M, g) and $x \in M$, there is a $\delta = \delta(M, g, x) > 0$ such that the metric g may be deformed to a metric $\hat{g}$ within $B_x(\delta)$ so that $\hat{g}$ is the constant curvature metric of scalar curvature s_g in $B_x(\delta^2)$ and such that the scalar curvature of $\hat{g}$ is arbitrarily close, (depending only on δ), to the constant s_g. Such deformations are local, in the sense that $\hat{g} = g$ outside $B_x(\delta)$, and further small, in the sense that $|\hat{g} - g|_{C^o} \leq \delta' = \delta'(\delta)$ in $B_x(\delta)$. A proof of this fact is given for instance in [Kb1, Lemma 3.2].

As an example, for any $\varepsilon > 0$, this glueing construction may be applied to the manifold $(N\#M, \bar{g}_{\varepsilon})$ constructed above; thus, one may choose $\varepsilon_1 << \varepsilon$ and glue another manifold N_1 into a small ε_1-ball in $(N, \bar{g}_{\varepsilon}) \subset (N\#M, \bar{g}_{\varepsilon})$ for instance, giving Yamabe metrics on $N_1\#N\#M$. Again, this process may be repeated inductively. Thus, one produces examples with curvature going to infinity at different scales or rates.

These variations of the construction have all blow-up limits which are either super-trivial or trivial solutions of the static vacuum equations, c.f. Remark 6.8 for further discussion; (note the one possible exception there however). Hence the geometry of blow-up limits near the locus $\{u = 0\}$ can be quite arbitrary, in contrast to the geometric structure of blow-ups obtained when the hypotheses of Theorem A(II) are satisfied. The only common structure of the blow-up limits here is that they are scalar-flat and asymptotically flat.

§6.2. Example 2. Let (M, g_o) be as in Example 1. Here, we will construct degenerating sequences of Yamabe metrics on connected sums, whose blow-up limits are the (doubled) Schwarzschild metric.

As indicated in the discussion in §3.1, static Einstein metrics are closely related to solutions, possibly only locally defined, of the equation $L^*u = 0$. Consider then static Einstein manifolds of the form $X = \Omega \times_h S^1$, with scalar curvature -12. Analogous to the (scalar-flat) Schwarzschild metric, consider the metric g on $\Omega = \mathbb{R}^+ \times S^2$, given by

$$g = dt^2 + f^2(t) ds^2_{S^2}, \tag{6.15}$$

where f is the solution to the following initial value problem; $f(0) = a > 0$, $f' \geq 0$,

$$(f')^2 = 1 + f^2 - (a^3 + a) f^{-1}, \tag{6.16}$$

see [Bes, Ch. 9J]. The scalar curvature of g is -6. Setting $h = f'$, one computes that

$$L^*h = 0, \tag{6.17}$$

and X is Einstein, with $Ric_X = -3 \cdot g_X$ by Proposition 3.0. The 2-sphere $S = \{t = 0\} = \{f = a\} = \{h = 0\}$ is totally geodesic and of constant curvature a^{-2}, while the metric g is asymptotic to the hyperbolic metric $H^3(-1)$. In fact, as $a \to 0$, the metric $g = g_a$ converges to the hyperbolic metric $H^3(-1)$ off S, while S is being crushed to a point. The curvature goes to infinity in small neighborhoods of S, and if one rescales g_a by the L^2 curvature radius, the blow-ups converge to the Schwarzschild metric.

Thus, as in Example 1, choose a small ball $B_x(\varepsilon) \subset (M, g_o)$. The smallest choice for ε is $\varepsilon \sim a^{1/3}$. For any $\varepsilon \geq a^{1/3}$, the neighborhood $N_{\varepsilon/2,\varepsilon} = \{x : \mathrm{dist}(x, S) \in (\varepsilon/2, \varepsilon)\}$ of S in (Ω, g_a) has uniformly bounded curvature as $a \to 0$, and for $\varepsilon >> a^{1/3}$ is almost isometric to the hyperbolic annulus $A(\varepsilon/2, \varepsilon) \subset B_x(\varepsilon)$. As in Example 1, we assume for simplicity that $\varepsilon >> a^{1/3}$, but $\varepsilon \to 0$ as $a \to 0$. Thus, by a small smooth perturbation, one may glue on $N_{0,\varepsilon} = I \times S^2$ with metric g_a onto $A(\varepsilon/2, \varepsilon)$ to obtain a metric g_a on $M \setminus$ (3-ball), with scalar curvature almost -6, and with totally geodesic, constant curvature boundary $S = S^2(a)$.

This process may be performed on any other hyperbolic manifold (M', g_o), and by matching at the isometric boundaries S, one obtains a smooth metric g_a on $M'\#M$, with scalar curvature almost -6, and with g_a isometric to g_o outside the ε -neighborhood of S. The metric g_a may then be conformally deformed to a Yamabe metric $\bar{g}_a$. One proves as in (6.10)-(6.12) that $\bar{g}_a$ is almost isometric to g_a; the conformal factor w converges to 1 in the C^o topology as $a \to 0$.

Here, the limit of $(M'\#M, \bar{g}_a)$ as $a \to 0$ is the 1-point union, at x, of the hyperbolic manifolds M and M'. The limit of blow-ups by the L^2 curvature radius at x, or at points where $h_a = \frac{1}{2}$ for example, is the Schwarzschild metric (0.17) doubled isometrically across the event horizon.

Consider briefly the behavior of the function $h = h_a$, as $a \to 0$. Clearly, on any fixed interval $[t_o, \mu)$, for $t_o > 0$, the functions h_a converge smoothly to the function $\cosh t > 1$. However, from (6.16), $h_a(0) = 0$, $h_a'(0) = a^{-1}$, so that h_a increases very rapidly from 0 to 1; compare with Propositions 3.18 or 3.19. In particular, the analogue of the descent construction of Theorem 3.10, with the function h_a in place of u_i, gives rise to blow-ups limits given by the Schwarzschild metric with limit potential function $h = (1 - 2mt^{-1})^{1/2}$, c.f. (0.17).

However, for the metrics $\bar{g}_a$ on $M'\#M$, there is no non-trivial descent down the u-levels, that is the function u does not satisfy the hypothesis (iii) of Theorem A(II), or Proposition 3.18/3.19. In fact, the limit function u is identically 0 on the (doubled) Schwarzschild blow-up limit, giving a super-trivial solution as in Example 1.

To see this, if u were non-zero on the Schwarzschild blow up, one must have

$$u = c \cdot h, \tag{6.18}$$

for some constant $c \neq 0$. (Since the limit is the Schwarzschild metric, as noted in Example 3.9 and Remark 5.6, the descent construction in Theorem 3.10 terminates at levels where u_a is bounded away from 0, so that the renormalization to $\bar{u}_a$ is not necessary).

Now, by Proposition 2.9, Δu is uniformly bounded in L^2 (as $a \to 0$). Thus,

$$\Delta u \to 0 \qquad \text{in } L^2, \tag{6.19}$$

in the blow-ups, (as in the proof of Proposition 3.1). Thus, in passing from M' into M through S, the function u must change sign; in the blow-up Schwarzschild limit, u is *odd* w.r.t. reflection in the core S, and is asymptotic to $-c$ or $+c$ at either end of the doubled Schwarzschild metric, corresponding to a region in M' or M respectively. This implies that the functions $u = u_a$ for the metrics $\bar{g}_a$ converge to a limit function u on the hyperbolic manifolds $M' \setminus \{x\}$ and $M \setminus \{x\}$ with

$$u(y) \to -c \qquad \text{as } y \to x \text{ in } M', \text{while} \tag{6.20}$$

$$u(y) \to +c \qquad \text{as } y \to x \text{ in } M.$$

It turns out that the behavior (6.20) is never possible when $c \neq 0$, c.f. Remark 6.8, although the proof is non-trivial. We prove the impossibility of (6.20) here in a simple special case, namely when $M = M'$. In this situation, there is an isometry ι of $(M\#M, \bar{g}_a)$ defined by reflection in the core 2-sphere $S = S^2(a)$. Let $u'_a = u_a \circ \iota$. From the invariance of the L^2 metric (1.3), the L^2 orthogonality of L^*u and ξ, and the invariance of $\bar{g}_a$ under ι, it follows easily that $L^*(u_a - u'_a) = 0$ on $(M\#M, \bar{g}_a)$. But Ker $L^* = 0$, (c.f. the beginning of §2), so that $u_a = u'_a$. Thus, u_a is an even function w.r.t. reflection in S. This property passes to the limit, (or the blow-up limit), and hence shows (6.20) is impossible.

Remark 6.2. **(i)** With only minor changes, this construction can be performed more generally on any pair of manifolds M, N, with Yamabe metrics of negative and equal scalar curvatures, using the local deformation to hyperbolic metrics in Remark 6.1(iii). We point out that both the parameters ε and a must be chosen sufficiently small, (in addition to satisfying $a^{1/3} << \varepsilon$), depending on the choice of Yamabe metrics on M and N.

To compare with Example 1, suppose g is any Yamabe metric on a manifold M with scalar curvature -6 say. Note that any manifold N with $\sigma(N) > 0$ carries unit volume Yamabe metrics with scalar curvature $-\delta$, for any $\delta > 0$. Such a metric may be rescaled so that the scalar curvature is -6 , so that it then has volume on the order of $\delta^{3/2}$ and diameter on the order of $\delta^{1/2}$. The construction in Example 2 may then be carried out on the pair M and N, (here ε will be much smaller than δ). This gives rise to a sequence of Yamabe metrics $\{g_i\}$ on $N\#M$, which converge smoothly on $M \setminus \{pt\}$ to g_o, while crushing N to a point. Thus, this sequence has the same basic features as the sequence in Example 1, except that blow-up limits are given by the isometrically doubled Schwarzschild metric. The triviality of the limit potential function u is discussed in Remark 6.8.

(ii). In fact, the construction in Example 2 is more general in most respects than that of Example 1, since one may form in addition connected sums with all manifolds N satisfying $\sigma(N) \geq 0$, (for example all graph manifolds). Namely, any such N has unit volume Yamabe metrics with scalar curvature $-\delta$. These may be scaled to make the scalar curvature -6 and volume $\sim \delta^{3/2}$ and the construction on $N\#M$ proceeds as above. When $\sigma(N) = 0$, the diameter of these rescaled metrics may remain large however, so that N may no longer be crushed to a point, but be (volume) collapsed, away from $\{pt\}$, to a possibly arbitrarily long lower dimensional space. (The volume collapse may not necessarily be with uniformly bounded curvature).

(iii). Similarly, all the constructions mentioned in Remark 6.1 can also be recaptured by modifications of the construction in Example 2, as above, except the construction in Remark 6.1(ii). Regarding this construction, we note that there are no smooth and complete static vacuum solutions which have more than 2 ends; for instance there is no complete static vacuum solution on a k-punctured 3-sphere, with $k \geq 3$. The Schwarzschild metric is a complete, conformally flat and scalar-flat metric on a 2-punctured 3-sphere.

§6.3. In this subsection, we analyse in somewhat greater detail the structure of the degeneration in Examples 1 and 2, in particular the structure induced on the limit (singular) manifold.

Throughout this subsection, let

$$M_o = \bigcup_1^n M_k \tag{6.21}$$

be a finite collection of connected closed hyperbolic manifolds, identified at a point x, with hyperbolic metric g_o. Let $\{g_i\}$ be a sequence of Yamabe metrics on a closed connected manifold M, converging in the Gromov-Hausdorff topology to (M_o, g_o) and converging smoothly to the hyperbolic metric g_o on $M_o \setminus \{x\}$. For example, M may be the connected sum of the components M_k, or M may be of the form $M = N \# M_o$, where $M_o \setminus \{x\}$ is connected, (i.e. $n = 1$), and $\sigma(N) > 0$, so that the sequence $\{g_i\}$ crushes N to the point $\{x\}$.

We also assume throughout §6.3, (until Remark 6.8 at the end), that $\{g_i\}$ has no blow-ups which are non-trivial solutions to the static vacuum equations with potential $u \not\equiv 0$. By the various constructions in §6.1 and §6.2, such sequences exist at least for many configurations M_o.

From Theorem 2.10, it is clear that the L^2 norm of z^T is uniformly bounded for the sequence $\{g_i\}$, and hence by (2.35), λ is bounded away from 0 and so $\delta = 1 - \lambda$ is bounded away from 1. Further, from Proposition 4.2 combined with Theorem A, the function $f = f_i$ does not converge to any constant in L^2. In particular we see that, for some subsequence,

$$\delta = \delta_i \to \delta_o > 0. \tag{6.22}$$

(If $\delta_i \to 0$, then from (2.35), $z^T \to 0$ in L^2, so that from the proof of Proposition 4.2, it follows that $u \to 1$ or $f \to 0$ in L^2, which is impossible by the statement above).

The convergence of the metrics g_i to the limit g_o is smooth away from x. Hence for instance the trace-less Ricci curvature z_i converges smoothly to the limit $z = 0$ away from x. Since f_i is uniformly bounded in $T^{2,2}(M, g_i)$ by Proposition 2.9, f_i converges at least weakly in $T^{2,2}$ away from x to a limit function $f \in T^{2,2}(M_o, g_o)$, while ξ_i converges weakly in L^2 to a limit $\xi \in L^2(M_o, g_o)$. In fact, the convergence of f_i and ξ_i to their limits is smooth away from x, as one sees by applying elliptic regularity to the equation (2.29). Since $\Delta f \in L^2(M_o \setminus \{x\}, g_o)$, and each component $(M_k \setminus \{x\}, g_o)$ of $(M_o \setminus \{x\}, g_o)$ extends smoothly to the hyperbolic metric on M_k, elliptic theory implies that f is bounded in $L^{2,2}$ and hence is a $C^{1/2}$ function on each M_k, c.f. [GT, Thm.8.12].

The next result shows that f extends continuously through x on M_o.

Lemma 6.3. *On the limit $(M_o \setminus \{x\}, g_o)$, we have*

$$f(y) \to -1 \qquad \text{as } y \to x, \tag{6.23}$$

in any component M_k of $M_o \setminus \{x\}$.

Proof. To see this, first note that since the curvature of $\{g_i\}$ blows up near x, Theorem 3.3 implies that u_i goes to 0 somewhere near x. Since there are no non-trivial blow-up limit solutions, Proposition 3.18 and Theorem 3.10 imply that for any $\varepsilon > 0$, with $B_i = (B_x(\varepsilon), g_i)$,

$$\operatorname*{osc}_{B_i} u_i \leq \delta = \delta(\varepsilon), \tag{6.24}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. (Note also that T_i is uniformly bounded by the remark following the proof of Proposition 4.2). It follows that (6.24) holds on the limit (M_o, g_o), which gives (6.23).

■

The basic identity (2.6), that is

$$z_i = L^* f_i + \xi_i,$$

on (M, g_i), passes to the limit and gives the equation

$$0 = L^* f + \xi, \tag{6.25}$$

on $(M_o \setminus \{x\}, g_o)$.

Of course there are no non-trivial smooth solutions to (6.25) on all of (M_o, g_o), since the terms are L^2 orthogonal. However, this is not the case in the presence of the singular point $\{x\}$.

Lemma 6.4. *The terms L^*f and ξ in (6.25) are not identically 0 on each component $M_k \setminus \{x\}$ of $M_o \setminus \{x\}$, so that the equation (6.25) is non-trivial on each M_k.*

Proof. Suppose $\xi = 0$ on $M_k \setminus \{x\}$ for some k. The trace equation (2.12), valid for $f = f_i$ on (M, g_i), passes to the limit away from $\{x\}$, so that we would then have

$$2\Delta f + sf = 0,$$

on $M_k \setminus \{x\}$. Since f is bounded, a standard (Riemann) removable singularity result implies that f extends smoothly over $\{x\}$ to give a smooth solution to this equation on M_k. However, since $s = -6 < 0$, the operator $-\Delta - \frac{s}{2}$ is positive, and thus one must have $f \equiv 0$ on M_k. This contradicts Lemma 6.3.

∎

In particular, on each $(M_k \setminus \{x\}, g_o)$, we have a non-trivial solution of the trace equation

$$2\Delta f + sf = \mathrm{tr}\ \xi. \tag{6.26}$$

One sees easily that if ξ were bounded in a neighborhood of $\{x\}$ in $(M_k \setminus \{x\}, g_o)$, then (6.25) holds weakly across $\{x\}$, which implies that ξ vanishes; more precisely, one uses a simple cutoff function argument to prove this. Thus ξ is unbounded near $\{x\}$, as is tr ξ, on each component $M_k \setminus \{x\}$.

To analyse the behavior of tr ξ, since $L(\xi) = 0$, $\delta\xi = 0$ and $z = 0$ on $(M_o \setminus \{x\}, g_o)$, we have from the defining equation (2.2)

$$\Delta\, \mathrm{tr}\ \xi + \frac{s}{3}\mathrm{tr}\ \xi = 0. \tag{6.27}$$

Exactly as in the proof of the non-triviality of (6.25), note that this equation has no bounded non-zero (weak) solutions on compact manifolds. Since tr $\xi \in L^2(M_o \setminus \{x\})$, and tr ξ is smooth away from x, it follows that, (up to a multiplicative constant), the unique non-zero solution to (6.27) is given by the Green's function $G_x(y)$ for the positive operator $-\Delta - \frac{s}{3} = -\Delta + 2$.

Thus, tr ξ is asymptotically of the form $c_k/t,\ t = \mathrm{dist}_{g_o}(\cdot, x)$, as $t \to 0$ on each M_k, i.e.

$$t \cdot \mathrm{tr}\ \xi \to c_k, \tag{6.28}$$

for some constant $c_k \neq 0$. The next result evaluates the constant c_k, c.f. also Lemma 2.3.

Lemma 6.5. *On each component $M_k \setminus \{x\}$ of $M_o \setminus \{x\}$, we have*

$$\int_{M_k} |\xi|^2 = -\frac{s}{3}\int_{M_k} \mathrm{tr}\ \xi = 4\pi c_k > 0. \tag{6.29}$$

Proof. Let $\eta = \eta_k$ be a cutoff function with $\eta \equiv 1$ on $M_k \setminus B_x(\varepsilon)$, $\eta \equiv 0$ on $B_x(\varepsilon/2)$, for ε small. Multiplying the equation (6.25) by $\eta\xi$ gives

$$\int \eta|\xi|^2 = \int \langle L^*f, \eta\xi\rangle = \int fL(\eta\xi) = f(x)\int L(\eta\xi) + O(\varepsilon), \tag{6.30}$$

where $O(\varepsilon) \to 0$ as $\varepsilon \to 0$. The last equality follows from Lemma 6.3. As above in (6.27), from (2.2) and the facts that $L(\xi) = 0$ and $\delta\xi = 0$, one computes

$$L(\eta\xi) = -\operatorname{tr}\xi\Delta\eta - 2\langle d\operatorname{tr}\xi, d\eta\rangle + \langle D^2\eta, \xi\rangle. \tag{6.31}$$

Thus, the divergence theorem gives

$$\int L(\eta\xi) = \int \eta\Delta\operatorname{tr}\xi + \int\langle d\eta, \delta\xi\rangle = \int \eta\Delta\operatorname{tr}\xi = -\frac{s}{3}\int \eta\operatorname{tr}\xi, \tag{6.32}$$

where the last inequality follows from (6.27). Hence, combining (6.30) and (6.32) and letting $\varepsilon \to 0$ gives

$$\int_{M_k} |\xi|^2 = -\frac{f(x)s}{3}\int_{M_k} \operatorname{tr}\xi,$$

and the first equality in (6.29) follows from Lemma 6.3.

For the second equality, apply the divergence theorem to (6.27) to obtain

$$-\frac{s}{3}\int_{M_k\setminus B_x(\varepsilon)} \operatorname{tr}\xi = \int_{S_x(\varepsilon)} \langle d\operatorname{tr}\xi, \nu\rangle,$$

where ν is the outward unit normal. Since $\nu = -\frac{d}{dt}$, use (6.28) to obtain

$$\lim_{\varepsilon\to 0}\int_{S_x(\varepsilon)} \langle d\operatorname{tr}\xi, \nu\rangle = 4\pi c_k.$$

■

The maximum principle applied to (6.27) implies that $\operatorname{tr}\xi$ does not change sign on any component $M_k \setminus \{x\}$ of $M_o \setminus \{x\}$. Hence, from Lemma 6.4, on each M_k,

$$\operatorname{tr}\xi > 0. \tag{6.33}$$

Returning to (6.26), it follows from Lemma 6.3 and (6.28) that near x on $M_k \setminus \{x\}$, f has the expansion

$$f = -1 + a_k t + o(t), \quad \text{as } t \to 0, \tag{6.34}$$

where $a_k = c_k/4 > 0$. In particular, f extends to a Lipschitz function on each (M_k, g_o).

Finally, from (6.25), we have

$$LL^*f = 0, \tag{6.35}$$

on $(M_o \setminus \{x\}, g_o)$. Since $z = 0$ on $M_o \setminus \{x\}$, from (2.27), (6.35) is of the form

$$2\Delta\Delta f + \frac{5}{3}s\Delta f + \frac{s^2}{3}f = 0. \tag{6.36}$$

It is easy to see, again from standard removable singularity results, that there are no non-zero C^2 solutions to (6.36) on M_k, so that f does not extend to a C^2 function on M_k. Observe that since $\operatorname{tr}\xi > 0$ by (6.33), the maximum principle applied to the trace equation (6.26) implies

$$f < 0 \qquad \text{on } M_o. \tag{6.37}$$

Recall that $\operatorname{tr}\xi$ is, (up to a multiplicative constant), the positive Green's function or fundamental solution of the operator $\Delta + \frac{s}{3}$ on M_k, with pole at x. Analogously for f, we have:

Proposition 6.6. *The function f satisfying (6.35) and (6.34) is, up to a multiplicative constant, the fundamental solution of the elliptic operator LL^* on M_k, with singularity at $\{x\}$.*

Further, the function f is the unique solution of (6.35) on M_o, smooth on $M_o \setminus \{x\}$ with $|f(y)+1| = O(t)$ as $t \to 0$, where $t(y) = dist(y, x)$ in M_o.

Proof. The leading order term of LL^* is the bi-Laplacian Δ^2, whose fundamental solution $F_x(p)$ based at x is asymptotic to $c \cdot t(p) = c \cdot \mathrm{dist}(x, p)$, for some $c \neq 0$. Since f is smooth away from $\{x\}$, this implies the first statement.

The second statement follows from the fact that the only solution of the equation $LL^*h = 0$, with $|h| = O(t)$, as $t \to 0$, is $h \equiv 0$. To see this, standard elliptic regularity applied to the equation (6.36) away from x implies that

$$|\Delta h| \cdot t^2 = o(t) \text{ and } |\nabla h| = O(t),$$

as $t \to 0$. Now pair (6.36) with $\eta \cdot h$, where η is a cutoff function as in Lemma 6.5, with $\mathrm{supp}|\nabla\eta| \subset A(\varepsilon, 2\varepsilon)$, $|\nabla\eta| \leq c/\varepsilon$, $|D^2\eta| \leq c/\varepsilon^2$. Integrating by parts gives

$$\int \eta(\Delta h)^2 + \eta|s||\nabla h|^2 + \eta s^2 h^2 = -\int \Delta h[h\Delta\eta + 2 < \nabla h, \nabla\eta >] + s\int h < \nabla h, \nabla\eta >.$$

The estimates above on h and its derivatives imply that the right side of this equation goes to 0, as $\varepsilon \to 0$. Since all terms on the left are non-negative, it follows that $h = 0$.

■

Since the function f exists on M_o, Proposition 6.6 implies that (conversely), independent of how f was constructed as a limit of $\{f_i\}$ from the geometry of (M, g_i), on any component M_k of M_o there exists a unique solution to (6.35) or (6.36), smooth away from x, which approaches the value -1 at x linearly in t. Such a solution has the expansion (6.34), and hence the constant $a_k = c_k/4 > 0$ in (6.34) is an invariant of the punctured manifold (M_k, x). It might be considered as a kind of mass for the operator LL^* at $\{x\}$. It follows that the terms L^*f and ξ in (6.25) exist and are determined by the geometry of each (M_k, x), independent of the approximating Yamabe sequence.

The fact that ξ and f are non-zero on the limit $(M_o \setminus \{x\}, g_o)$ shows that they detect the 'bad' convergence of $\{g_i\}$ near the singular point $\{x\}$, even on the limit. If $\{\gamma_i\}$ is a sequence of Yamabe metrics on a connected hyperbolic manifold $M_o = M$, converging smoothly to (M_o, g_o) everywhere, then of course $\xi \to 0$, $z^T \to 0$ and $f \to 0$ smoothly. When $M \neq M_o$, it is obvious though that the metric g_o itself extends smoothly over $\{x\}$ to give the hyperbolic metric on the closed manifold M_o. This is related to the following consequence of Lemma 6.5.

Corollary 6.7. *For (M, g_i) as above, we have*

$$\lim_{i\to\infty} \int_{M_k} |\xi_i|^2 dV_{g_i} = \int_{M_k} |\xi|^2 dV_{g_o} \tag{6.38}$$

and

$$\lim_{i\to\infty} \int_{M_k} |L^*u_i|^2 dV_{g_i} = \int_{M_k} |L^*u|^2 dV_{g_o}. \tag{6.39}$$

Proof. Theorem 2.2, (2.10), and the continuity of the scalar curvature and volume in the convergence to the limit (M_o, g_o) show that (6.38) and (6.39) are equivalent. Fatou's theorem, (lower semi-continuity), implies that, for each k,

$$\int_{M_k} |\xi|^2 dV_{g_o} = \int_{M_k\setminus\{x\}} |\xi|^2 dV_{g_o} \leq \liminf_{i\to\infty} \int_{M_k\setminus\{x\}} |\xi_i|^2 dV_{g_i}, \tag{6.40}$$

where for the integral on the right we consider $M_k \setminus \{x\} \subset M$ via a Gromov-Hausdorff approximation.

Summing over k then gives

$$\int_{M_o} |\xi|^2 dV_{g_o} \leq \liminf_{i\to\infty} \int_M |\xi_i|^2 dV_{g_i} = \liminf_{i\to\infty} -\frac{s}{3}\int_M \mathrm{tr}\ \xi_i\ dV_{g_i} = -\frac{s}{3}\int_{M_o} \mathrm{tr}\ \xi\ dV_{g_o} = \int_{M_o} |\xi|^2 dV_{g_o}.$$

Here, the first equality uses (2.14), the second uses the fact that $\xi_i \to \xi$ weakly in L^2 and the third follows from Lemma 6.5. It follows that equality holds in (6.40) and the lim inf may be replaced by limit.

■

Of course the L^2 norm of $L^* f_i$ and of z_i diverges to ∞ as $i \to \infty$. In other words, Corollary 6.6 shows that the splitting of the metric g_i in (2.10) is continuous in the limit $i \to \infty$, but the splitting of the curvature z_i is far from continuous.

As a curiosity at this stage, we note that an easy computation, using the relation (6.29), implies

$$\frac{d}{dt}(v^{2/3} \cdot s)(g_o + t\xi)|_{t=0} = -v^{-1/3} \int_{M_o\setminus\{x\}} |\xi|^2 dV. \tag{6.41}$$

Thus deforming the metric g_o on $M_o \setminus \{x\}$ in the direction $-\xi$ increases the Yamabe constant $v^{2/3} \cdot s$ of g_o. Of course since ξ is unbounded, this is not a smooth perturbation of the hyperbolic metric on M_o.

Remark 6.8. In this remark, we consider briefly the converse of the previous discussion in §6.1-§6.3, namely to what extent blow-up limits of Yamabe metrics converging to a singular hyperbolic limit as above *must* be super-trivial solutions of the static vacuum equations.

Thus, let (M, g_i) be any sequence of Yamabe metrics converging to $M_o \cup X$, where M_o is as in (6.21) and X, (possibly empty) is a lower dimensional space of zero volume, compare with Remark 6.2(ii). Suppose further that the blow-up limit at the singular point $x \in M_o$ is a complete, smooth, non-flat and asymptotically flat manifold (N, g_b), i.e. all ends are asymptotically flat. All of the constructions and their variations in §6.1 and §6.2 satisfy these assumptions, (assuming the construction is not iterated).

By Theorem A(I), the blow-up limit (N, g_b) is a solution of the static vacuum equations, which is non-flat; (x is essentially the point of maximal curvature concentration). Hence, for instance by Theorem 1.1, either (N, g_b) is a super-trivial solution or it is the (isometrically doubled) Schwarzschild metric, with non-vanishing potential function u. Thus, we are interested in considering only if the latter possibility occurs.

If the blow-up limit (N, g_b) is the Schwarzschild metric, then of course one must have $M = M_1 \# M_2$, with the Schwarzschild neck $S^2 \times \mathbb{R}$ joining the two factors. This situation occurs in all versions of the construction in Example 2 in §6.2. However, for the construction in §6.1, it occurs

only in the situation of Remark 6.1(ii), where a sum of two Green's functions is used w.r.t. the conformal Laplacian on the standard (S^3, g_{can}).

We may assume that M_2 is hyperbolic, and M_1 is either hyperbolic or is a manifold satisfying $\sigma(M_1) \geq 0$. In the former case, M_o is the one point union at x of M_1 and M_2, while in the latter case, $M_o = M_2$ and M_1 is collapsed to X. (Note that one may have $X = \{x\}$; for instance, this is necessarily the case for the construction in Remark 6.1(ii), which requires $N = M_1$ with $\sigma(M_1) > 0$).

It turns out that if M_1 is hyperbolic, then the Schwarzschild neck (N, g_b) must be a super-trivial solution, for any sequence $\{g_i\}$ satisfying the hypotheses above. On the other hand, if M_1 is any graph manifold, (in particular $\sigma(M_1) \geq 0$), and $\{g_i\}$ collapses M_1 with bounded curvature away from the neck to X, then the limit solution is non-trivial. The limit potential function u is given by the Schwarzschild potential, odd under reflection through the event horizon, as in (6.20). The construction in Theorem 3.10 produces this non-trivial blow-up limit.

For the remaining possible situations, where $\sigma(M_1)$ is not a graph manifold, or $\{g_i\}$ volume collapses M_1 to a lower dimensional space, but the collapse is with unbounded curvature away from the neck, the triviality or non-triviality of the solutions is unknown. (Note this situation includes the construction in §6.1, Remark 6.1(ii)).

We do not give a complete proof of these statements here, since essentially no new ideas are needed, and since these statements will not be used. Instead, we just sketch the ideas involved.

When M_1 is a graph manifold and $\{g_i\}$ collapses M_1 with bounded curvature, this statement can be deduced quite easily from the work in §6.5, in particular from Proposition 6.9. In this case, in fact $\xi \to 0$, $z^T \to 0$ and $u \to 1$ in $L^2(M, g_i)$; c.f. Remark 6.10(ii). Note that M_1 becomes invisible, in terms of volume and total scalar curvature, in $\{(M, g_i)\}$.

On the other hand, if M_1 is assumed hyperbolic, then u cannot converge to 1 in L^2, essentially for the reasons discussed following (6.19). One may then prove that u_i is approximately 0 for i large on the Schwarzschild neck by using the minimizing property Proposition 2.7 for L^*u, together with the existence of non-trivial solutions ξ, L^*f to (6.25), (6.27) and (6.34)-(6.35) on $M_o \setminus \{x\}$, as discussed following Proposition 6.6.

§6.4. Example 3. In this next example, we let $M = S^2 \times S^1$, and consider a maximizing family $\{g_i\}$ of Yamabe metrics on M with $\mathrm{vol}_{g_i} M = \mathrm{vol}\, S^3(1)$. It is known that

$$\sigma(S^2 \times S^1) = \sigma(S^3). \tag{6.42}$$

In fact, see [Kb1,2], [Sc2], there are conformally flat, spherically (i.e. S^2) symmetric Yamabe metrics g_i, thus of the form

$$g_i = dt^2 + f_i^2(t) ds_{S^2}^2, \tag{6.43}$$

with $s_{g_i} \to \sigma(S^3)$. As noted in [La], and as in Example 2, setting $h_i = f_i'$, one computes

$$L^*(h_i) = 0, \tag{6.44}$$

w.r.t. the metrics g_i globally on M. In fact, c.f. [La],

$$\mathrm{Ker}\, L^* =< h_i >,$$

on (M, g_i). Again from Proposition 3.0, the 4-manifolds $X = M \times_{h_i} S^1$ are Einstein, with scalar curvature $s_X \to 12$, the scalar curvature of $S^4(1)$. (These metrics have cone singularities along two totally geodesic 2-spheres in X, corresponding to the locus where $h_i = 0$).

As $i \to \infty$, the manifolds $(S^2 \times S^1, g_i)$ converge to the space (Z, g_o), where $Z = S^3/\{p\} \sim \{-p\}$ is the 3-sphere with two antipodal points identified, and g_o is the canonical metric of constant curvature and volume 1, see [An1], [Sc2]. The convergence is smooth on the complement of the point $\{p\} \sim \{-p\}$, and the functions h_i, for an appropriate normalization, converge to an eigenfunction h of the Laplacian on S^3, with eigenvalue 3, with $h_i(p) \to 1$, $h_i(-p) \to -1$.

Now consider the equation (2.10) on (M, g_i), i.e.

$$L^* u + \xi = -\frac{s}{3} g. \tag{6.45}$$

The function $u = u_i$ is only determined up to functions in Ker L^*. Using suitable multiples of the functions h_i in (6.44), we may arrange that

$$||u_i||_{L^2} \to \infty, \qquad \text{as } i \to \infty. \tag{6.46}$$

(We note that it may well be necessary to use the functions h_i in this way; there may be representatives $u' \in \{u + \text{Ker } L^*\}$ which are uniformly bounded in L^2). In case (6.46) holds, as discussed in §3.2-§3.4, the construction of the buffered blow-up in Theorem 3.10 requires that $u = u_i$ be renormalized by its maximum. Thus, setting $\bar{u} = \frac{u}{\sup u}$, we obtain from (6.45) that

$$L^* \bar{u}_i \to 0. \tag{6.47}$$

Then in fact $\bar{u}_i$ approaches the function h_i in (6.44) and $\{\bar{u}_i\}$ limits on the eigenfunction h of the Laplacian as above. If one performs the descent construction of Theorem 3.10 on $\{\bar{u}_i\}$, one obtains as blow-up limit the Schwarzschild metric (0.17) doubled across the totally geodesic boundary S^2. In the blow-up, the functions $\bar{u}_i$ limit on the potential function u of the Schwarzschild metric, and the harmonic function u is odd w.r.t reflection in S^2. At one end, $u \to +1$ while $u \to -1$ at the other end, corresponding to the two points {p} and $\{-p\}$ respectively in the limit (Z, g_o).

Thus in this simple example, the blow-up limit obtained from Theorem A(II) mirrors exactly the degeneration of the sequence (M, g_i).

§6.5. Example 4. The discussion in §6.1 and §6.2 raises the question of what are the simplest kinds of degenerations of Yamabe metrics which *are* modeled on non-trivial solutions of the static vacuum equations, (leaving Example 3 aside). Thus, let M_1 and M_2 be closed hyperbolic 3-manifolds and $M = M_1 \# M_2$. Based on a modification of Example 2, we construct a sequence $\{g_i\}$ of Yamabe metrics on M which crush the essential 2-sphere in M, converge smoothly almost everywhere w.r.t. volume to the hyperbolic metrics on M_1 and M_2, and for which blow-up limits obtained via Theorem 3.10 are non-trivial solutions to the static vacuum equations. It is interesting to compare the construction below with the work of O. Kobayashi in [Kb1, Thm.2].

First, return to the construction in Example 2 on each manifold M_k, $k = 1, 2$. This gives a metric g_a on $M_k \setminus B_{x_k}(a)$, with totally geodesic, constant curvature boundary $S = S^2(a)$, which is hyperbolic outside the ball $B_{x_k}(\varepsilon)$ and is "Schwarzschild-like" in the neck $A(a, \varepsilon/2)$. As in Examples 1,2 note that the curvature of g_a is uniformly bounded in $A(\varepsilon/2, \varepsilon)$ provided $\varepsilon \geq a^{1/3}$. As previously, we assume $\varepsilon >> a^{1/3}$, so that the sectional curvature of g_a is very close, (depending only on $\varepsilon/a^{1/3}$) to -1 on $A(\varepsilon/2, \varepsilon)$, and the scalar curvature of g_a is almost -6 everywhere.

Now instead of identifying $M_k \setminus B_{x_k}(a)$ along their isometric boundaries, we extend g_a past $S^2(a)$ with the metric g on $S^2 \times \mathbb{R}^+$ in (6.15). The resulting metric, still called g_a, is a complete metric on $\tilde{M}_k \equiv M_k \setminus B_{x_k}(a) \cup S^2 \times \mathbb{R}^+$. Note that this metric is conformally equivalent, in a

natural way, to a metric on $M_k \setminus \{x_k\}$ which extends smoothly over $\{x_k\}$, so that $(\tilde{M}_k \cup \{x_k\}, g_a)$ is conformally compact. (Of course $(M_k \setminus B_{x_k}(a), g_a)$ cannot be conformally compactified by adding a point). The blow-up limit of $(\tilde{M}_k, g_a)$ at or near x_k as $a \to 0$ is the complete, isometrically doubled Schwarzschild metric. The end of $(\tilde{M}_k, g_a)$ is asymptotically hyperbolic. In fact, just as before on the other 'outer' side of $S^2(a)$ in $\tilde{M}_k$, the 'inner' annulus $A_i = A(\varepsilon/2, \varepsilon) \subset (S^2 \times \mathbb{R}^+, g_a)$ is almost isometric to an annulus $A_{-1}(\varepsilon/2, \varepsilon)$ in $H^3(-1)$ for $\varepsilon >> a^{1/3}$. In particular, the curvature of g_a is bounded in both the inner annulus $A_i \subset S^2 \times \mathbb{R}^+$ and the (isometric) 'outer' annulus $A_o \subset M_k$.

To join the two inner annuli A_i, we choose a metric γ_a on $G_a \equiv S^3 \setminus \ (B_1 \cup B_2)$, B_j a 3-ball, so that γ_a is isometric to (A_i, g_a) near its boundary. Thus, the metrics $(\tilde{M}_1, g_a)$ and $(\tilde{M}_2, g_a)$ can be glued along $(S^3 \setminus B_1 \cup B_2), \gamma_a)$ to give a smooth family of metrics, still denoted g_a, on $M = \tilde{M}_1 \cup G_a \cup \tilde{M}_2 = M_1 \# M_2$. The gluing metric γ_a is chosen to have curvature z_{γ_a} uniformly bounded as $a \to 0$, scalar curvature $s_{\gamma_a} \to -6$ everywhere as $a \to 0$, and to be *volume collapsing,* so that

$$\mathrm{vol}_{\gamma_a} G_a \to 0, \quad \text{as } a \to 0. \tag{6.48}$$

Of course since the metrics g_a are not collapsing near the boundaries ∂B_i of G_a, the collapse of γ_a takes place on a scale much larger than ε. We sketch the construction of such metrics.

Construction of glueing metrics: View S^3 as the union of two solid tori $D^2 \times S^1$, glued along the torus boundary T^2 by interchanging the two circles in T^2. Consider a complete warped product metric on $D^2 \times S^1$ of the form

$$h = g_{D^2} + f^2 d\theta^2, \tag{6.49}$$

where f is a positive function on D^2. We assume that outside a compact set, h is isometric to a rank 2 hyperbolic cusp, so that in particular $f = f(r) = e^{-r}$ for r large.

It is a standard fact that h may be chosen to satisfy

$$\int s_h \, dV_h < 0, \tag{6.50}$$

c.f. [Bes, Thm.4.32] for example. Now let $\bar{h}$ be a Yamabe metric conformal to h, so that $\bar{h} = \psi^4 \cdot h$. It is also quite standard, (c.f. [AM, Thm.C] for a proof), that (6.50) and the assumptions on the asymptotic form of h imply that $\bar{h}$ exists and is complete, with ψ uniformly bounded away from 0 and ∞. Possibly after a rescaling, the scalar curvature $\bar{s}$ of $\bar{h}$ satisfies $\bar{s} = -6$.

Now we claim that the metric $\bar{h}$ is also a warped product, i.e. is invariant under the S^1 action on $D^2 \times S^1$. To see this, let $F = F_\theta$ be an isometry of h, (from the S^1 family). We have

$$F^*\bar{h} = (F^*\psi)^4 \cdot (F^*h) = (F^*\psi)^4 \cdot h = v^4 \cdot \bar{h},$$

where $v = (F^*\psi)/\psi$ is a bounded function. The metrics $F^*\bar{h}$ and $\bar{h}$ are thus conformal Yamabe metrics, of the same scalar curvature -6 and volume. Hence v satisfies the equation (1.12), i.e.

$$-6v^5 = -8\Delta v - 6v,$$

w.r.t. the $\bar{h}$ metric. The maximum principle then implies that $v \equiv 1$. Thus, F is also an isometry of $\bar{h}$.

It follows that ψ is a function on D^2 and the defining equation (1.12) becomes, on D^2,

$$8\Delta_{D^2}\psi + 8 < \nabla\psi \, , \, \nabla log f >= 6\psi^5 + s\psi. \tag{6.51}$$

Since ψ is bounded and $s = -6$ outside a compact set, the maximum principle applied to (6.51) shows that $\psi(x) \to 1$ as $x \to \infty$ in $D^2 \times S^1$.

It follows that $\bar{h}$ is also of the form (6.49), with f replaced by $\bar{f}$, and is asymptotic to a rank 2 hyperbolic cusp at infinity.

Now observe that the full curvature of $\bar{h}$ is unchanged when the length of the S^1 factor is changed. In fact, for any $\alpha > 0$ (small), the metrics

$$\bar{h}_\alpha = \bar{g}_{D^2} + (\alpha\bar{f})^2\, d\theta^2, \tag{6.52}$$

are locally isometric to $\bar{h}$. Note that the length of the S^1 fiber is asymptotic to αe^{-r}.

Next, for R sufficiently large depending on the choice of α, on the annulus $A(R, 2R)$, we may bend the metric $\bar{g}_{D^2} \sim dr^2 + e^{-2r} d\phi^2$ slowly on $A(R, 2R)$ so that on $A(2R-1, 2R)$, it has the form $dr^2 + (\alpha\bar{f})^2\, d\phi^2$. Here we assume that both θ and ϕ are parameters in $[0, 2\pi]$. The resulting metric $\tilde{h}_\alpha$ has uniformly bounded curvature, independent of α and R, scalar curvature arbitrarily close, depending only on R, to -6, and is almost isometric to a rank 2 hyperbolic cusp on $A(2R-1, 2R)$.

Finally, on the annulus $A(2R, 4R)$, the function $\alpha\bar{f}$ is again bent slowly so that at the boundary $S(4R)$, $\alpha\bar{f}(4R) \sim \alpha e^{-4R}$, $\bar{f}'(4R) = 0$ and $\bar{f}$ extends smoothly as an even function under reflection through $4R$. Again this may be done so that the curvature is uniformly bounded and the scalar curvature is arbitrarily close to -6. Call the resulting metric on $D^2 \times S^1$ again $\tilde{h}_\alpha$.

Now the boundary $S(4R)$ is a totally geodesic, flat, and square torus T^2, in the sense that the generators $d\phi$ and $d\theta$ are orthogonal and of the same length. Hence the metric $\tilde{h}_\alpha$ may be doubled across the boundary T^2, with an interchange of the S^1 factors, to give a smooth metric on S^3. This is the metric γ_a, where α in (6.52), essentially the maximal length of the S^1 fiber in (6.52), is relabeled to $a = a(\alpha)$, the size of the core S^2 in (M_k, g_a) above. We require $a << \alpha$, in fact $\varepsilon << a$, but $a \to 0$ implies $\alpha \to 0$.

By choosing $R = R_\alpha$ suitably, the resulting family of smooth metrics γ_a on S^3 has scalar curvature converging smoothly to -6 as $a \to 0$, has uniformly bounded curvature, and is volume collapsing in the sense of (6.48). Observe that this volume collapse requires

$$\mathrm{diam}_{\gamma_a} S^3 \sim 2R \to \infty, \qquad \text{as } a \to 0. \tag{6.53}$$

Now the inner annuli $(A_i(\varepsilon/2, \varepsilon), g_a)$ in each $\tilde{M}_k$ are almost isometric to hyperbolic ε-annuli. For $\varepsilon << \alpha$, the metrics γ_a on the geodesic ball $B_p(\varepsilon)$ contained in each $D^2 \times S^1 \subset S^3$ are fixed, independent of a or α; here $p = (p_o, \theta)$, where p_o is the 'center' of D^2. (The metric on D^2 is fixed, only the length of the fiber S^1 changes with a). Hence, just as with the first or outer glueing as in Example 2, the metric γ_a may be smoothly matched to (A_i, g_a), keeping the curvature uniformly bounded and the scalar curvature arbitrarily close to -6, (c.f. the local deformation in Remark 6.1(iii)). This completes the construction of the glueing metrics.

Finally, as in Examples 1, 2, let $\bar{g}_a$ be the Yamabe metric conformal to g_a with the same volume on M, so that $\bar{g}_a = w^4 \cdot g_a$. Since the scalar curvature of g_a approaches -6 everywhere as $a \to 0$, as in (6.12), $w \to 1$ pointwise as $a \to 0$. Thus, the metric $\bar{g}_a$ is C^o close to g_a. Further, from the defining equation (1.12), $|\Delta w| \to 0$ pointwise as $a \to 0$. Using L^2 estimates for the elliptic equation (1.12) on any geodesic ball B about q, of radius $\rho(q)$, one obtains w.r.t. the g_a metric,

$$\int_B |D^2 w|^2 << \max\Big(1, \int_B |z|^2\Big) \quad \text{as } a \to 0, \tag{6.54}$$

provided $\rho(q) \leq 1$. Of course $\bar{g}_a$ is smoothly close to g_a on $M \setminus B_{x_1}(\varepsilon) \cup B_{x_2}(\varepsilon))$. This completes the construction of the Yamabe metrics $\{\bar{g}_a\}$ on M.

The justification for this construction comes from the following result.

Proposition 6.9. *On the family* $(M, \bar{g}_a)$,

$$||z^T||_{L^2} \to 0 \qquad \text{as } a \to 0. \tag{6.55}$$

In particular, $\xi \to 0$ *in* L^2, *and* $\delta \to 0$.

From Proposition 4.1 for instance, it follows that

$$u = u_a \to 1 \qquad \text{in } L^2, \tag{6.56}$$

and hence, (c.f. Proposition 4.2), one may carry out the descent construction in Theorem 3.10 to obtain non-trivial blow-up limit solutions of the static vacuum equations. In fact for an initial sequence of points $\{x_i\}$ in either $M_k \setminus B_{x_k}(a)$, with $x_i \to x_k$, $u_i(x_i) \to 1$, and $a = a(i) \to 0$, the maximal blow-up limit (N, g_o), c.f. §5.1, based at a buffered sequence $\{y_i\}$ obtained by u-descent from $\{x_i\}$ is the complete doubled Schwarzschild metric. The limit potential function is $u = \pm(1 - \frac{2m}{t})^{1/2}$, c.f. (0.17).

Proof of Proposition 6.9. To prove (6.55), we use the characterization (2.41) of $||z^T||_{L^2}$, i.e.

$$\int_M |z^T|^2 = \inf_{\int \phi = 0} \int_M |z - L^*\phi|^2. \tag{6.57}$$

Of course (6.57) is taken w.r.t. the $\bar{g}_a$ metric. However, using the fact that $\bar{g}_a$ is C^o close to g_a, and smoothly close away from $M \setminus (B_{x_1}(\varepsilon) \cup B_{x_2}(\varepsilon))$, together with (6.54), it suffices to estimate the right hand side of (6.57) w.r.t. the g_a metric. To see this, refer to the expansion (2.42). Using standard formulas, c.f. [Bes, Ch.1J], under the conformal change from g_a to $\bar{g}_a$, all terms in (2.42) differ by arbitrarily small quantities as $a \to 0$, except the term containing $|z|^2$, which differs by a term on the order of $|D^2 w|^2$, where w is the conformal factor. By (6.54), this is small in $B_{x_1}(\varepsilon) \cup B_{x_2}(\varepsilon)$ compared with the dominant $|z|^2$ term in (2.42).

Of course, since the metric g_a is quite explicit, it is easier to estimate (6.57) w.r.t. g_a than w.r.t. $\bar{g}_a$. All the estimates to follow are then on (M, g_a).

The function $\phi = \phi_a$ is defined on the various pieces of (M, g_a) as follows. First, on each $M_k \setminus B_{x_k}(\varepsilon)$, since $z = 0$, we set $\phi \equiv 0$, so that

$$\int_{M_k \setminus B_{x_k}(\varepsilon)} |z - L^*\phi|^2 = 0. \tag{6.58}$$

Next, on the (doubled) Schwarzschild-like neck N_a joining the inner and outer annuli $A(\varepsilon/2, \varepsilon)$ on each $\tilde{M}_k$, let

$$\phi = h - 1, \tag{6.59}$$

where $h = h_a$ is the function from (6.16)-(6.17). Note that h is odd under reflection through the core $S = S^2(a)$ in each M_k. Since $L^*h = 0$ w.r.t. the metric g_a, $L^*\phi = r$ on N_a. Hence, since $\mathrm{vol}\, N_a \to 0$ as $a \to 0$, and the scalar curvature is uniformly bounded,

$$\int_{N_a} |z - L^*\phi|^2 \to 0, \quad \text{as } a \to 0. \tag{6.60}$$

To examine the behavior on the outer or first glueing annulus $A_o = A(\varepsilon/2, \varepsilon) \subset M_k$, the metric g_a has uniformly bounded curvature on A_o as $a \to 0$. Now from the discussion in Example 2, in A_o,

$$h \sim \cosh t = 1 + \frac{1}{2}t^2 + o(t^2), \tag{6.61}$$

so that $\phi \sim 0$. Further, since $h \sim h(t)$,

$$|D^2 h| \sim |h' D^2 t + h''(dt)^2| \leq C, \tag{6.62}$$

uniformly in A_o as $a \to 0$, by (6.61). Hence the function ϕ may be smoothed in A_o so that $\phi \equiv 0$ near the outer boundary $S(\varepsilon)$ of A_o, keeping $D^2\phi$ uniformly bounded. Since $\mathrm{vol}\, A_o \to 0$ as $a \to 0$, it follows that (6.60) holds also over A_o.

Finally, in the glueing region G_a joining $\tilde{M}_1$ and $\tilde{M}_2$, and containing the inner annuli A_i, the curvature of g_a is uniformly bounded while $\mathrm{vol}\, G_a \to 0$, so that

$$\int_{G_a} |z - L^*\phi|^2 \leq \delta_a + 2\int_{G_a} |L^*\phi|^2 \leq \delta_a + c\int_{G_a} (|D^2\phi|^2 + \phi^2), \tag{6.63}$$

where $\delta_a \to 0$ as $a \to 0$. Now in the inner annulus $A_i = A(\varepsilon/2, \varepsilon) \subset S^2 \times \mathbb{R}^+$, by considerations similar to (6.61), $\phi \sim -2$, while as in (6.62), $|D^2\phi|$ is uniformly bounded. Hence, using (6.53), we may extend ϕ from a neighborhood of both components of ∂G_a into G_a - for instance as a function of a suitable smoothing of the distance function to x_1 (or x_2) - so that ϕ and $D^2\phi$ remain uniformly bounded and so that, (most importantly),

$$\int_M \phi dV_{g_a} \sim \int_M \phi dV_{\bar{g}_a} = 0. \tag{6.64}$$

Again, since $\mathrm{vol}\, G_a \to 0$, it follows that the right side of (6.63) goes to 0 as $a \to 0$. These estimates together then prove (6.55).

■

The limit of $(M, \bar{g}_a)$ as $a \to 0$ is the union of the two hyperbolic manifolds M_1 and M_2, glued along a lower dimensional length space of infinite diameter and zero volume. In particular, M_1 and M_2 are infinitely far apart in the limit. The pointed Hausdorff limits of $(M, \bar{g}_a, q_a)$ as $a \to 0$ are complete subsets of this structure, based at some limit point $q = \lim q_a$.

Of course, the limit here is quite different from the limits of the sequences in §6.1-§6.3.

Remark 6.10. **(i)** Observe that in obtaining (6.55), rather strong use has been made of the fact that the limit metrics on M_1 and M_2 are hyperbolic. Although the construction in Example 2 is valid quite generally, (c.f. Remark 6.2(i)), and does not require the limits to be hyperbolic, (6.55) will no longer hold for constructions as above with non-hyperbolic limits.

(ii). In addition to taking connected sums of hyperbolic manifolds, one may carry out this construction on manifolds of the form $N\#M$, where N is any closed oriented graph manifold, (and hence $\sigma(N) \geq 0$), compare with Remark 6.2(ii).

Namely, carry out the construction above on $M_2 = M$ and replace the glueing region $G_a = S^3 \setminus (B_1 \cup B_2)$ by $N \setminus B$. Just as S^3 admits metrics γ_a satisfying (6.48) and the curvature conditions preceding (6.48), so does the manifold N. The remainder of the argument then proceeds as before. Proposition 6.9 holds as before also.

As in Remarks 6.1 and 6.2, since the construction is essentially local, it can be iterated (arbitrarily) many times.

Remark 6.11. It is worthwhile to mention explicitly that all known constructions or examples of degenerating sequences of Yamabe metrics on a given 3-manifold M have blow-up limits which do have one common feature; namely, they are all scalar-flat and asymptotically flat.

7. Palais-Smale Sequences for Scalar Curvature Functionals.

The discussion through §5 applies to quite general sequences of Yamabe metrics, and has made no assumptions that the sequence $\{g_i\}$ is a maximizing sequence for $\mathcal{S}|_{\mathcal{C}}$ or even a sequence for which $\mathcal{S}(g_i)$ approaches a critical value of $\mathcal{S}|_{\mathcal{C}}$. Of course, we are most interested in understanding the degeneration of a maximizing sequence of Yamabe metrics, and one would expect that such a sequence may have some restrictions on its behavior not valid for general sequences.

In this section, we examine more closely the structure of the space of metrics $\mathbb{M}$ and its completeness properties w.r.t. natural metrics. This is then used to understand the existence of Palais-Smale sequences for natural functionals on $\mathbb{M}$ or its subvarieties. In particular, the differences in the behaviors of the examples in §6 can be understood from this viewpoint.

§7.1. The space $\mathbb{M}$ is, of course, not complete with respect to the L^2 or $L^{2,2}$ metrics. We study its completion w.r.t. the $L^{2,2}$ norm, (and later the $T^{2,2}$ norm). The $L^{2,2}$ norm (1.4) on the collection $\{T_g\mathbb{M}\}$ of tangent spaces generates a length metric $L^{2,2}$ on $\mathbb{M}$, by defining $L^{2,2}(g_1, g_2)$ to be the infimum of the lengths of curves joining g_1 to g_2.

Let $\mathbb{M}_{L^{2,2}}$ be the Cauchy-completion of $\mathbb{M}$ w.r.t. the $L^{2,2}$ metric, so an (ideal) point in $\mathbb{M}_{L^{2,2}}$ is a limit point of a Cauchy sequence w.r.t. the $L^{2,2}$ metric. Let

$$\bar{\mathbb{M}} = \mathbb{M}_{L^{2,2}} \cap C^0(\mathbb{M}), \tag{7.1}$$

where $C^0(\mathbb{M})$ is the space of continuous Riemannian metrics on M.

By work in [E], the space $\bar{\mathbb{M}}$ may be identified with the space of Riemannian metrics on M which, with respect to a fixed smooth coordinate atlas, have local expressions which are $L^{2,2}$ functions of the local coordinates. The tangent space $T_g\bar{\mathbb{M}}$ is naturally identified with the Hilbert space of symmetric bilinear forms on M, locally in $L^{2,2}$, with inner product given as in (1.4).

Let $\mathbb{M}_o \subset \mathbb{M}$ be the space of metrics having a fixed volume form dV, of total volume 1, and $\mathbb{M}_1 \subset \mathbb{M}$ the space of metrics of total volume 1. Similarly, let $\bar{\mathbb{M}}_o$, and $\bar{\mathbb{M}}_1$ be the intersection of the $L^{2,2}$-completion of $\mathbb{M}_o$ or $\mathbb{M}_1$ with $C^0(\mathbb{M}_o)$ or $C^0(\mathbb{M}_1)$ respectively. The spaces $\bar{\mathbb{M}}$, $\bar{\mathbb{M}}_o$ and $\bar{\mathbb{M}}_1$ are infinite dimensional Hilbert manifolds in the topology generated by the $L^{2,2}$ norm, c.f. [E].

It is important to note that the spaces $\bar{\mathbb{M}}$, $\bar{\mathbb{M}}_o$, $\bar{\mathbb{M}}_1$ need not apriori be (Cauchy) complete in the $L^{2,2}$ metric. By definition, of course $\mathbb{M}_{L^{2,2}}$ is complete. However, the symmetric bilinear forms in such a completion may not be positive definite, and thus not metrics. Thus, the restriction that $\bar{\mathbb{M}}$ be contained in $C^0(\mathbb{M})$ implies that $\bar{\mathbb{M}}$ might not apriori be complete.

To understand the completeness of these spaces, following the discussion in [E], we first decompose the space of smooth metrics $\mathbb{M}$ as

$$\mathbb{M} = Vol(M) \times \mathbb{M}_o, \quad g = \phi \cdot h \tag{7.2}$$

where $Vol(M)$ is the space of volume forms on M, h is an arbitrary metric in $\mathbb{M}_o$ and ϕ is the ratio of the volume forms of g and h to the power $2/n = 2/3$. Note that the metric g in (7.2) is conformal to h, so that $\mathbb{M}_o$ gives a representation of the space of conformal structures on M. This should be compared with the splitting of $\mathbb{M}$ into the space of conformal classes and the space $\mathcal{C}$

of Yamabe metrics. Here we recall a well-known result of Moser [Mo] that any metric in $\mathbb{M}_1$ may be pulled back by a diffeomorphism of M to a metric in $\mathbb{M}_o$. In particular, for any unit volume Yamabe metric $g \in \mathcal{C}_1$, there is a diffeomorphism ψ of M such that $\psi^* g \in \mathbb{M}_o$.

Given a fixed background metric $g_o \in \mathbb{M}_o$, any metric $h \in \mathbb{M}_o$ may be written as

$$h = g_o \cdot e^X, \tag{7.3}$$

where X is trace-free w.r.t. g_o; here we are using the metric g_o to identify bilinear forms with linear maps.

Lemma 7.1. *The space $\bar{\mathbb{M}}_o$ is (Cauchy) complete.*

Proof. Let $\{g_i\}$ be a Cauchy sequence in $\bar{\mathbb{M}}_o$ w.r.t. the $L^{2,2}$ metric. Then by definition, c.f. (7.1), the sequence $\{g_i\}$ converges to an element $g \in \mathbb{M}_{L^{2,2}}$, i.e. a symmetric bilinear form on M with local component functions in the $L^{2,2}$-completion of $C^\infty(M)$. By Sobolev embedding $g_i \to g$ in the C^0 topology. Since the volume forms dV_{g_i} are fixed, it follows that $dV_g = dV_{g_i}$. In particular, dV_g is a (continuous) positive 3-form on M. However, the symmetric bilinear forms in $\mathbb{M}_{L^{2,2}}$ are necessarily positive semi-definite. It follows that g is a positive definite form and hence in $\bar{\mathbb{M}}_o$.

∎

It follows from the Hopf-Rinow theorem in infinite dimensions that $\bar{\mathbb{M}}_o$ is also geodesically complete w.r.t. the $L^{2,2}$ metric, i.e. $L^{2,2}$ geodesics in $\bar{\mathbb{M}}_o$ exist for all time. We note that $\mathbb{M}_o$ itself is geodesically complete, (but of course not Cauchy-complete), w.r.t the L^2 metric, c.f. [E, Thm.8.9]; in fact the L^2 geodesics in $\mathbb{M}_o$ are given by the simple expression, c.f. (7.3),

$$g(t) = g \cdot e^{tA} \subset \mathbb{M}_o, \tag{7.4}$$

where $A \in T_g\mathbb{M}_o$, so that A is a smooth symmetric bilinear form, trace-free w.r.t. g. In particular, these geodesics exist for all time. Further, the L^2 exponential map is a diffeomorphism onto $\mathbb{M}_o$.

On the other hand, the full space $\bar{\mathbb{M}}$ is not complete with respect to the $L^{2,2}$ metric, nor geodesically complete w.r.t. the L^2 metric. With regard to the latter, in contrast to (7.4), the geodesics of $\mathbb{M}$ with respect to the L^2 metric do not exist for all time, but may leave the space $\mathbb{M}$ in finite time. In [FG], the L^2 geodesic of $\mathbb{M}$, with initial position (μ, g) and initial velocity (w, A), c.f. (7.3) and (7.4), is calculated to be given by

$$g(t) = (v(t)^2 + n^2t^2)^{2/3} g \cdot \exp\left(\frac{\tan^{-1}(nt/v)}{n} A\right), \tag{7.5}$$

where $v(t) = 1 + \frac{1}{2}(\frac{w}{\mu})t$ and $n = \frac{1}{4}(3\,\mathrm{tr}\ A^2)^{1/2}$.

A brief inspection, as noted in [FG], shows that these curves escape from $\mathbb{M}$ in finite time, (i.e. the form $g(t)$ is no longer positive definite), if the initial velocity matrix A vanishes somewhere on M. For example, in the volume or conformal directions, when $A \equiv 0$, it is apparent from (7.5) that $g(t)$ becomes degenerate, i.e. non-positive definite, in finite time.

Although we will not carry it out here, it is not difficult to verify that there are curves $g(t)$, $t \in [0, 1)$, of the form (7.5), with $A \equiv 0$, of finite length in the $L^{2,2}$, which degenerate as $t \to 1$. In particular, the L^2 norm of the curvature blows up as $t \to 1$.

On the other hand, if the matrix A never vanishes on M, i.e., if $A(p) \neq 0$, for all $p \in M$, then the geodesic (7.5) with initial velocity (w, A) continues in $\mathbb{M}$ for all time, for any initial metric (μ, g). Thus, in most all directions in $\bar{\mathbb{M}}$, the L^2 geodesic (7.5) exists for all time.

Given this background, we now discuss the main result of this section. Let $\mathcal{F} : \bar{\mathbb{M}}_o \to \mathbb{R}^+$ be a C^1 functional on $\bar{\mathbb{M}}_o$, which is thus bounded below, We only consider 'natural' functionals, in the sense that $\mathcal{F}(\phi^* g) = \mathcal{F}(g)$, for any $\phi \in \mathrm{Diff}(M)$. For example, it is easy to see that the total scalar curvature functional $\mathcal{S}$, or the L^p norm of the scalar curvature $\mathcal{S}^p$, for $1 < p \leq 2$, extends to a C^1 functional on $\bar{\mathbb{M}}_o$ or $\bar{\mathbb{M}}_1$.

Theorem 7.2. *Suppose $\mathcal{F} : \bar{\mathbb{M}}_o \to \mathbb{R}_+$ is a C^1 natural functional, as above, and let $\{\gamma_i\} \in \mathbb{M}_o$ be a minimizing sequence for $\mathcal{F}$. Given any $\varepsilon > 0$, there is another minimizing sequence $\{g_i\} \in \mathbb{M}_o$ for $\mathcal{F}$, with $L^{2,2}(g_i, \gamma_i) \leq \varepsilon$, such that $\nabla\mathcal{F}(g_i) \to 0$, as $i \to \infty$ in the dual $(L^{2,2})^*$ norm, i.e.*

$$|\nabla\mathcal{F}|^*(g_i) \equiv \sup_{|\alpha|_{L^{2,2}=1}} \left| \int_M \langle \nabla_{g_i}\mathcal{F}, \alpha \rangle_{g_i} dV_{g_i} \right| \to 0, \qquad \text{as } i \to \infty, \tag{7.6}$$

for $\alpha = \alpha_i \in T_{g_i}\bar{\mathbb{M}}_o$.

Proof. For otherwise, given any $\varepsilon > 0$, there exists $c = c(\varepsilon) > 0$ such that, (for some subsequence),

$$|\nabla\mathcal{F}|^*(g_i) \geq c, \qquad \text{as } i \to \infty. \tag{7.7}$$

for all $g_i \in \bar{\mathbb{M}}_o$ such that

$$L^{2,2}(g_i, \gamma_i) \leq \varepsilon. \tag{7.8}$$

Let $\gamma_i(t)$ be a smooth curve in $\bar{\mathbb{M}}_o$, with $\gamma_i(0) = \gamma_i$, $|\frac{d}{dt}\gamma_i(t)| = 1$, (i.e. $\gamma_i(t)$ is parameterized by arclength in the $L^{2,2}$ metric), and

$$\int_M \langle \nabla_{\gamma_i(t)}\mathcal{F}, \frac{d}{dt}\gamma_i(t) \rangle dV_{\gamma_i(t)} \leq -\frac{c}{2}. \tag{7.9}$$

Since $\bar{\mathbb{M}}_o$ is an infinite dimensional manifold and $\nabla\mathcal{F}$ is continuous, it is obvious from (7.7) that such curves exist for t sufficiently small, say for $0 < t \leq t_o = t_o(i)$; one could take for instance $\gamma_i(t)$ to be the L^2 geodesic (7.4) in the direction $-\nabla_{\gamma_i}\mathcal{F}$, but reparametrized w.r.t. $L^{2,2}$ arclength.

If $t_o < \varepsilon$, we just repeat the process above starting at $\gamma_i(t_o)$. Since the $L^{2,2}$ metric is complete on $\bar{\mathbb{M}}_o$, it is clear that one obtains in this way a continuous piecewise smooth curve $\gamma_i(t)$ in $\bar{\mathbb{M}}_o$, satisfying (7.9) for $0 \leq t \leq \varepsilon$. By definition, (7.9) then gives

$$\frac{d}{dt}\mathcal{F}(\gamma_i(t)) = \int_M \langle \nabla_{\gamma_i(t)}\mathcal{F}, \frac{d}{dt}\gamma_i(t) \rangle dV_{\gamma_i(t)} \leq -\frac{c}{2}. \tag{7.10}$$

for all $t \in [0, \varepsilon]$. By integration of (7.10), it follows that

$$\mathcal{F}(\gamma_i(t)) \leq \mathcal{F}(\gamma_i) - \frac{c}{2} \cdot t. \tag{7.11}$$

The fact that (7.11) is valid for all $t \leq \varepsilon$, contradicts the fact that $\{\gamma_i\}$ is a minimizing sequence for $\mathcal{F}$. Hence (7.7) cannot hold in conjunction with (7.8) and the result follows.

■

A sequence of metrics $\{g_i\}$ satisfying (7.6) will be called a *Palais-Smale* sequence for $\mathcal{F}$, w.r.t. the $L^{2,2}$ norm. Note that this definition corresponds to one of the hypotheses in the well-known Condition C of Palais-Smale [PS]. Of course, the condition (7.6) should not be confused with Condition C—a compactness condition which is much too strong to be of any use in this context.

Thus Theorem 7.2 implies that within any ε-neighborhood, w.r.t. the $L^{2,2}$ metric, of a minimizing sequence for $\mathcal{F}$, there exists a Palais-Smale minimizing sequence.

Since, by Moser's theorem mentioned above, $\mathbb{M}_1$ is obtained from $\mathbb{M}_o$ by the action of the diffeomorphism group, and $\mathcal{F}$ together with (7.6) is diffeomorphism invariant, it follows that Theorem 7.2 holds for $\mathcal{F}$ considered as a functional $\mathcal{F} : \bar{\mathbb{M}}_1 \to \mathbb{R}^+$, with $\alpha \in T\mathbb{M}_1$.

Remark 7.3. It is easy to see that Theorem 7.2 also holds for functionals on other natural submanifolds of $\mathbb{M}_1$, obtained by diffeomorphims from $\mathbb{M}_o$. For instance, at least in case $\sigma(M) \leq 0$, so that $\mathcal{C}_1$ is an infinite dimensional submanifold of $\mathbb{M}_1$, one may verify without difficulty that Theorem 7.2 holds for $\mathcal{S}|_{\mathcal{C}_1}$, (or $-\mathcal{S}|_{\mathcal{C}_1}$).

§7.2. Since the $L^{2,2}$ metric is rather strong, the corresponding dual $(L^{2,2})^* = L^{-2,2}$ metric is rather weak. It is thus of interest to understand if the results above can be strengthened by use of a weaker norm than the $L^{2,2}$ norm. Of course, the main issue in §7.1 is the Cauchy completeness of $\bar{\mathbb{M}}_o$.

In this respect, the following might be useful.

Lemma 7.4. *The completion of* $\mathbb{M}$ *w.r.t. the* $T^{2,2}$ *norm (1.9), as in (7.1), is the same as its completion* $\bar{\mathbb{M}}$ *w.r.t. the* $L^{2,2}$ *norm.*

Proof. Let g_i be a $T^{2,2}$ Cauchy sequence in $\mathbb{M}$, converging to an element g in $\mathbb{M}_{T^{2,2}} \cap C^0(M)$. Thus g is a continuous metric on M, and the local components g_{kl} of g in a smooth atlas on M are $T_g^{2,2}$ functions of the local coordinates. It follows that $\Delta_g(g_{kl})$ is locally in L^2 on M. Hence, using the continuity of g, elliptic regularity theory, c.f. [GT, Thms.8.8,9.15], implies that g_{kl} is locally in $L_g^{2,2}$. Thus $g \in \bar{\mathbb{M}}$, which gives the result. ■

It is now easy to verify that all of the results above, in particular Theorem 7.2, hold also w.r.t. the weaker $T^{2,2}$ norm; hence there exist (many) Palais-Smale sequences for $\mathcal{F}$ as above w.r.t. the $T^{2,2}$ norm.

Let us apply this to the functional $\mathcal{S}|_{\mathcal{C}_1}$. If g_i is a Palais-Smale sequence of Yamabe metrics in $\mathcal{C}_1$, then

$$\|\nabla_{g_i}(\mathcal{S}|_{\mathcal{C}_1})\|_{T^{-2,2}(T\mathcal{C}_1)} = \sup_{|\alpha|_{T^{2,2}}=1} \left| \int_M \langle z^T, \alpha \rangle dV_{g_i} \right| \to 0, \tag{7.12}$$

where $z^T = z^T_{g_i}$ and $\alpha \in T_{g_i}\mathcal{C}_1$. Now (unfortunately), the condition that $\alpha = \alpha_i \in T_{g_i}\mathcal{C}_1$ is a global condition on α, (as a tensor on (M, g_i)), and hence it is difficult to obtain local information on z^T from (7.12). If (7.12) could be strengthened to hold for all $\alpha \in T_{g_i}\mathbb{M}_1$, i.e. if $\{g_i\}$ is strongly Palais-Smale in the sense of (4.6), then the estimate (7.12) does provide non-trivial local control. In particular, Propositions 4.1 and 4.2 hold in this situation. We do not know however if there exist such strongly Palais-Smale sequences on general 3-manifolds M.

With regard to the examples of §6, it is easily verified that all sequences in Example 4 are Palais-Smale for $\mathcal{S}|_{\mathcal{C}_1}$ w.r.t. the $T^{2,2}$ norm. In fact, such sequences are strongly Palais-Smale, in the sense of (4.6), since the full L^2 norm of z^T goes to 0 by Proposition 6.9. Note here that of course $||z^T||_{T^{-2,2}} \leq ||z^T||_{L^2}$ on $\{g_i\}$. Similarly, the sequences in Example 3 are strongly Palais-Smale, since the same, but in this case much simpler, reasoning proving Proposition 6.9 holds here also. On the other hand, Examples 1 and 2 are clearly not strongly Palais-Smale for $\mathcal{S}|_{\mathcal{C}_1}$ w.r.t. $T^{2,2}$, since for instance $\lim_{i\to\infty}(z^T)_{g_i}$ is non-zero on the limit (M_o, g_o), c.f. §6.3. It seems non-trivial to prove, although we certainly conjecture, that the sequences in Example 1 and 2 are not Palais-Smale for $\mathcal{S}|_{\mathcal{C}_1}$.

8. Appendix

In this Appendix, we prove Theorem 3.2. We break the proof into several parts. The first result states that the event horizon Σ must be non-empty.

Theorem A.1. *The only complete solution (N, g, u) to the static vacuum Einstein equations with $u > 0$ everywhere is the flat solution on $\mathbb{R}^3$ or a quotient $\mathbb{R}^3/\Gamma$, with $u = const.$*

Proof. Since u is positive on the complete manifold N, the associated 4-manifold $X^4 = N^3 \times_u S^1$ is Ricci-flat and complete, c.f. (1.14). Define the function h by

$$h = \log u.$$

A straightforward calculation gives

$$\Delta_X h = 0. \tag{A.1}$$

Suppose first that either u is bounded above, or bounded below away from 0. Thus, h has either a uniform upper or a uniform lower bound. The Liouville theorem of Yau [Yu] then implies that $h = const.$, and thus $u = const. > 0$. The equations then immediately imply that the metric g is flat.

In general, since X^4 is Ricci flat, the infinitesimal Harnack inequality of Cheng-Yau [CY, Theorem 6] gives the bound

$$\sup_{B^4(r)} |\nabla(\log v)| \leq C \cdot \frac{1}{r},$$

for any positive harmonic function v defined on a geodesic ball $B^4(r) \subset X$. In particular, by integration, this leads to the usual Harnack inequality

$$\sup_{B^4(r)} v \leq C \cdot \inf_{B^4(r)} v. \tag{A.2}$$

Now suppose u is not constant. Then applying the estimate (A.2) to the functions $h - a$ and $b - h$, where a and b, depending on r, are chosen to make these functions positive on $B^4(r)$ implies, by a well known technique due to Moser, c.f. [GT, Theorem 8.22], that u has oscillation growing at a definite power of r, as $r \to \infty$. In fact, an observation of Cheng [Cg] is that if h is non-constant, then h must have at least linear growth, i.e. there is a constant $C < \infty$ such that, for $r \geq 1$,

$$\operatorname*{osc}_{B^4(r)} h = \sup_{B^4(r)} h - \inf_{B^4(r)} h \geq C \cdot r, \tag{A.3}$$

where as above $B^4(r)$ is the geodesic r-ball about some fixed point $x_o \in X$. Suppose first that

$$\inf_{B^4(r)} h < -c_1 r, \tag{A.4}$$

for some $c_1 > 0$. It follows again from the Harnack inequality that for any r, there are points $x_r \in S^3_{\pi(x_o)}(r) \subset N$ such that $u(x) \leq c_2 e^{-c_1 \cdot r}$, for all $x \in B^3_{x_r}(1)$, r large, for some constant $c_2 > 0$; here $\pi : X^4 \to N^3$ is the projection on the first factor and $S^3(B^3)$ denote geodesic spheres (resp. balls) in N. We use this to estimate the volume of regions in X^4. Setting $U_{x_r}(1) = \pi^{-1}(B^3_{x_r}(1))$, we have

$$\mathrm{vol}_{X^4}\, U_{x_r}(1) = \int_{B_{x_r}(1)} u\ dV \leq c_3 \cdot \mathrm{vol}\, B^3_{x_r}(1) e^{-c_1 \cdot r}. \tag{A.5}$$

To estimate $\mathrm{vol}\, B^3_{x_r}(1)$, it follows from the Harnack inequality as above that there is a constant C_o, independent of r, s.t.

$$C_o^{-1} \leq \frac{\sup_B \ u(x)}{\inf_B \ u(x)} \leq C_o. \tag{A.6}$$

where we set $B = B^3_{x_r}(1) \subset U_{x_r}(1)$; here we are using the fact that u is invariant in the fiber or S^1 factor. Consider the conformally equivalent metric $\tilde{g} = u^2 g$ from (1.19). Thus, if $u_o = \inf_B u(x)$, then $\mathrm{vol}_g B \leq u_o^{-3} \ \mathrm{vol}_{\tilde{g}} B$. On the other hand, since distances in $\tilde{g}$ are at most $C_o u_o$ times distances in g, it follows that $B \subset B_{\tilde{g}}(C_o \cdot u_o)$. Since from (1.20), $Ric_{\tilde{g}} \geq 0$, the volume comparison theorem for Ricci curvature implies $\mathrm{vol}_{\tilde{g}} B(C_o \cdot u_o) \leq \frac{4}{3}\pi C_o^3 u_o^3$. It follows there is a constant D, independent of r, s.t.

$$\mathrm{vol}\, B^3_{x_r}(1) \leq D, \tag{A.7}$$

so that from (A.5), we obtain

$$\mathrm{vol}_{X^4}\, U_{x_r}(1) \leq c_4 e^{-c_1 \cdot r}, \tag{A.8}$$

for some constant c_4. However, (A.8) contradicts the (relative) volume comparison principle. Namely, for Ricci flat 4-manifolds, and $r \geq 1$, one has the bound

$$\frac{\mathrm{vol}\, B^4_{x_r}(r)}{r^4} \leq \mathrm{vol}\, B^4_{x_r}(1) = \mathrm{vol}\, U_{x_r}(1),$$

where the equality follows from the fact that the fiber S^1 has exponentially small length near $x_r \in N \subset X$. Together with (A.8), this implies

$$\mathrm{vol}\, B^4_{x_r}(r) \leq c_4 r^4 e^{-c_1 \cdot r} \to 0 \qquad \text{as } r \to \infty,$$

which is of course impossible.

On the other hand, if (A.4) does not hold, then by (A.3), one has the estimate

$$\sup_{B^4(r)} h > c_5 \cdot r,$$

for some $c_5 > 0$. Hence, there are points $x_r \in N \subset X$ near which the fiber S^1 has exponentially large growth. Arguing in exactly the same way as above, from the volume comparison theorem for non-negative Ricci curvature, $\mathrm{vol}\, B_{x_r}(1) \geq c_6 r^{-3}$ and hence,

$$\mathrm{vol}\, B^4_{x_r}(r) \geq c_7 r^{-3} e^{c_5 \cdot r},$$

which is again impossible by the relative volume comparison theorem on X.

It follows that h must be constant and thus (N, g) is flat.

$\blacksquare$

We note the following compactness principle for solutions of the static equations.

Lemma A.2. *Let (g_i, u_i) be a sequence of solutions to the static vacuum equations (0.16), defined on a geodesic ball $B_i = B_{x_i}(1)$, (w.r.t. the metric g_i). Suppose*

$$\mathrm{vol}\, B_i \geq v_o > 0, \qquad |r_i| \leq \Lambda < \infty, \tag{A.9}$$

and

$$u_i(x_i) \geq u_o > 0, \qquad u_i(y_i) > 0, \quad \forall y_i \in B_{x_i}(1), \tag{A.10}$$

for some constants v_o, Λ, h_o. *Then, for any* $d > 0$, *a subsequence of the Riemannian manifolds* $(B_{x_i}(1-d), g_i)$ *converges, in the* C^∞ *topology, modulo diffeomorphisms, to a limit manifold* $(B_x(1-d), g)$, *with limit function* u. *The triple* $(B_x(1-d), g, u)$ *is a smooth solution of the static vacuum equations.*

Proof. The Cheeger-Gromov compactness theorem, c.f. Theorem 1.3 and references there, implies that a subsequence of (B_i, g_i) converges, in the $C^{1,\alpha}$ topology, to a limit manifold (B, g). The bounds (A.9) and (A.10) imply further that the positive harmonic functions u_i satisfy a uniform Harnack inequality in $B_i(1-d)$. If there is a uniform bound $u_i(x_i) \leq u^o < \infty$, it follows that a subsequence of $\{u_i\}$ converges in the C^o topology to a limit positive function u. If not, so $u_i(x_i) \to \infty$, we renormalize u_i by setting $\bar{u}_i = u_i/u_i(x_i)$. Then again the fact that $\bar{u}_i$ is harmonic and the Harnack inequality imply that $\{\bar{u}_i\}$ has a subsequence converging to a positive limit harmonic function u in $B(1-d)$. As noted in §1.3, since the metrics g_i satisfy an elliptic equation (0.16), the regularity theory for elliptic equations implies that one has $C^{k,\alpha}$ bounds for solutions, in terms of $C^{1,\alpha}$ bounds, i.e. all covariant derivatives of the curvature are bounded in terms of the bounds (A.9)-(A.10). Equivalently, one can pass to the Ricci flat 4-manifold N and use the Einstein equation to obtain $C^{k,\alpha}$ regularity, for any k. These estimates imply convergence in the C^k topology, and thus C^∞ convergence.

Using elliptic regularity on the equations (0.16), the L^∞ bound on $|r|$ in (A.9) may be replaced by weaker bounds; for instance, it suffices to assume an L^2 bound on r.

■

The following Corollary gives an apriori estimate for the curvature of a solution away from the event horizon Σ. The proof is a standard consequence of Theorem A.1 and Lemma A.2.

Corollary A.3. *Let* (N, g, u) *be a solution to static vacuum equations (0.16), and* $U \subset N$ *a domain with smooth boundary on which* $u > 0$. *Then there is an (absolute) constant* $K < \infty$, *independent of* (N, g) *and* U, *such that for all* $x \in U$,

$$|z|(x) \leq \frac{K}{t(x)^2}, \tag{A.11}$$

where $t(x) = \mathrm{dist}(x, \partial U)$.

Proof. The proof is by contradiction. Thus, assume that (A.11) does not hold. Then there are static vacuum solutions (N_i, g_i, u_i), smooth domains $U_i \subset N_i$ on which $u_i > 0$ and points $x_i \in U_i$ such that

$$t^2(x_i) \cdot |z_i|(x_i) \to \infty, \quad \text{as } i \to \infty. \tag{A.12}$$

Let $t_i = t(x_i)$. Since it may not be possible to choose the points x_i so that they maximize $|z_i|$ (over large domains), we shift the base points x_i as follows: choose $s_i \in [0, t_i)$ such that

$$s_i^2 \sup_{B_{x_i}(t_i - s_i)} |z_i| = \sup_{s \in [0, t_i)} \big(s^2 \cdot \sup_{B_{x_i}(t_i - s)} |z_i|\big) \to \infty, \quad \text{as } i \to \infty, \tag{A.13}$$

where the last estimate follows from (A.12), (set $s_i = t_i$). Let $y_i \in B_{x_i}(t_i - s_i)$ be points such that

$$|z_i|(y_i) = \sup_{B_{x_i}(t_i - s_i)} |z_i|. \tag{A.14}$$

Further, setting $s = s_i(1 - \frac{1}{k}),\ k > 1$, in (A.13), one obtains the estimate

$$s_i^2|z_i|(y_i) \geq s_i^2 \left(1 - \tfrac{1}{k}\right)^2 \cdot \sup_{B_{x_i}(t_i - s_i(1-\frac{1}{k}))} |z_i| \geq s_i^2 \left(1 - \tfrac{1}{k}\right)^2 \cdot \sup_{B_{y_i}(s_i/k)} |z_i|, \tag{A.15}$$

so that

$$\sup_{B_{y_i}(s_i/k)} |z_i| \leq \left(1 - \tfrac{1}{k}\right)^{-2} |z_i|(y_i), \tag{A.16}$$

Now rescale or blow-up the metric so that $|\tilde{z}_i|(y_i) = 1$ by setting $\tilde{g}_i = |z_i|(y_i) \cdot g$, and consider the pointed sequence $(U_i, \tilde{g}_i, y_i)$. We have

$$|\tilde{z}_i|(y_i) = 1, \tag{A.17}$$

and by (A.13) and scale invariance,

$$\mathrm{dist}_{\tilde{g}_i}(y_i, \partial U_i) \to \infty, \qquad \text{as } i \to \infty. \tag{A.18}$$

Also, it follows from (A.16) that

$$|\tilde{z}_i|(x) \leq C(\mathrm{dist}_{\tilde{g}_i}(x, y_i)). \tag{A.19}$$

We also normalize u by setting

$$\tilde{u}_i(x) = \frac{u(x)}{u(y_i)}, \tag{A.20}$$

and note by construction that $\tilde{u}_i > 0$ on U_i.

Suppose first that there exists $v > 0$ such that

$$\mathrm{vol}_{\tilde{g}_i} B_{y_i}(1) \geq v. \tag{A.21}$$

Then the volume comparison principle for bounded curvature and (A.21) imply that $\mathrm{vol}_{\tilde{g}_i} B_{x_i}(1) \geq v(x_i)$, where $v(x_i)$ depends only on $\mathrm{dist}_{\tilde{g}_i}(x_i, y_i)$. By the compactness of solutions, Lemma A.2, it follows that a subsequence converges, in the C^∞ topology, to a limit solution $(U_\infty, g_\infty, u_\infty)$, which is complete and satisfies $u_\infty > 0$ everywhere. (The minimum principle for harmonic functions implies that u_∞ cannot vanish anywhere). By Theorem A.1, g_∞ must be flat and u_∞ constant. However, the smooth convergence guarantees that the equality (A.17) passes to the limit, contradicting the fact that g_∞ is flat.

If (A.21) is not satisfied, so that $\mathrm{vol}_{\tilde{g}_i} B_{y_i}(1) \to 0$, as $i \to \infty$, it follows that the sequence $(U_i, \tilde{g}_i, y_i)$ is collapsing in the sense of Cheeger-Gromov on balls $(B_{y_i}(R_i), \tilde{g}_i)$, where $R_i \to \infty$ as $i \to \infty$. In dimension 3, the structure theory of collapse implies that the collapse is along an injective F-structure. More precisely, the balls $(B_{y_i}(R_i), \tilde{g}_i)$ have the structure of a Seifert fibration, with fibers that are injective in the fundamental group, c.f. [An2, §2,3] or [An II, §2]. Thus, one may pass to the universal cover of $B_{y_i}(R_i)$. Since the universal covers no longer collapse, i.e. (A.21) is satisfied, one may apply the discussion above to the universal covers, and obtain a contradiction in the same manner.

∎

We note that exactly the same proof can be used to prove also that

$$|\nabla \log u|(x) \leq \frac{K}{t(x)}. \tag{A.22}$$

In particular, these results together prove Theorem 3.2.

Note that since K is independent of the domain U, (A.11) holds for t the distance to the event horizon Σ, (provided this is defined), even if Σ is singular. More generally but for the same reasons, (A.11) holds for t the distance to ∂N^o, where N^o is the maximal domain on which the potential $\bar{u}$ is positive, c.f. Theorem 5.1. This is because N^o is the Hausdorff limit of an exhaustion of N^o by smooth subdomains.

Remark A.4. (i). Similarly, using elliptic regularity associated to the static vacuum equations, one may show in the same way that for any $k \geq 1$,

$$|\nabla^k z|(x) \leq \frac{C(k)}{t^{2+k}(x)}, \qquad |D^k \log u|(x) \leq \frac{C(k)}{t^k(x)}. \tag{A.23}$$

(ii). Under the same hypotheses as Corollary A.3, suppose also that there is an end $E \subset N$ such that

$$u(x) \to const. > 0, \tag{A.24}$$

as $x \to \infty$ in E. Then essentially the same proof shows that there is a function $\mu = \mu(t)$ such that

$$|z|(x) \leq \frac{\mu(t(x))}{t^2(x)}, \qquad |\nabla \log u|(x) \leq \frac{\mu(t(x))}{t(x)}, \tag{A.25}$$

where $t(x)$ is the distance to a fixed base point in E. Namely, if (A.25) does not hold, then one derives the same estimates as (A.12), (A.13), (A.18), with ∞ replaced by some constant $c > 0$ and concludes the proof as before using the fact that static vacuum solutions with $u = const.$ are flat.

References

[Ak] K. Akutagawa, Yamabe metrics of positive scalar curvature and conformally flat manifolds, Diff. Geom. Appl., vol 4., (1994), 239-258.

[An1] M. Anderson, Scalar curvature and geometrization conjectures for 3-manifolds, Comparison Geometry, MSRI Publ., vol. 30, (1997), 49-82.

[An2] ______, Extrema of curvature functionals on the space of metrics on 3-manifolds, Calc. of Var. & P.D.E., vol. 5:3, (1997), 199-269.

[An3] ______, Convergence and rigidity of manifolds under Ricci curvature bounds, Inventiones Math., vol. 102, (1990), 429-445.

[An4] ______, On the structure of solutions to the static vacuum Einstein equations, (preprint, Stony Brook, July 1998).

[AnII] ______, Scalar curvature, metric degenerations and the static vacuum Einstein equations on 3-manifolds, II, (preprint, Stony Brook, December 1998). This and the papers above are available at: http://www.math.sunysb.edu/ ~ anderson/

[AC] M. Anderson and J. Cheeger, C^α compactness for manifolds with Ricci curvature and injectivity radius bounded below, Jour. Diff. Geom., vol. 35, (1992), 265-281.

[Au1] T. Aubin, Équations différentielles nonlinéares et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl., 55, (1976), 269-296.

[Au2] ______, The scalar curvature, Differential Geometry and Relativity, M. Cahen, M. Flato eds., Reidel Publ. Co., Dordrecht, Holland (1976).

[AM] P. Aviles and R. McOwen, Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds, Jour. Diff. Geom., vol. 27, (1988), 225-239.

[BE] M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, Jour. Diff. Geom., vol. 3, (1969), 379-392.

[Bes] A. Besse, Einstein Manifolds, Ergebnisse der Mathematik, 3. Folge, Band 10, Springer Verlag, New York (1987).

[Bo] J.P. Bourguignon, Une stratification de l'espace des structures Riemanniennes, Compositio Math., vol 30, (1975), 1-40.

[BM] G.Bunting and A. Masood-ul-Alam, Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-times, Gen. Rel. and Grav., vol 19, (1987), 147-154.

[C] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math., 92, (1970), 61-74.

[CG1] J. Cheeger and M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded I, Jour. Differential Geometry, vol. 23, (1986), 309-346.

[CG2] ——, Collapsing Riemannian manifolds while keeping their curvature bounded II, Jour. Differential Geometry, vol. 32., (1990), 269-298.

[Cg] S.Y. Cheng , Liouville theorem for harmonic maps, Proc. Symp. Pure Math., vol. 36, (1980), 147-151.

[CY] S.Y. Cheng and S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math., vol. 28, (1975), 333-354.

[DK] D. DeTurck and J. Kazdan, Some regularity theorems in Riemannian geometry, Ann. Sci. Ecole Norm. Sup., vol.14, (1981), 249-260.

[E] D. Ebin, The manifold of Riemannian metrics, Proc. Symp. Pure Math, A.M.S., vol 15, (1968), 11-40.

[FG] D. Freed and D. Groisser, The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomporphism group, Mich.Math.J., vol. 36, (1989), 323-344.

[GT] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of the Second Order, 2^{nd} Edition, Springer Verlag, New York, 1983.

[G] M. Gromov, Structures Métriques pour les Variétés Riemanniennes, Cedic-Fernand/Nathan, Paris (1981).

[HE] S. Hawking and G. Ellis, The Large Scale Structure of Space-Time, Cambridge U. Press, Cambridge, (1973).

[Ha] W. Hayman, Subharmonic Functions, Academic Press, London (1976).

[I] W. Israel, Event horizons in static vacuum space-times, Phys. Rev., vol 164:5, (1967), 1776-1779.

[Kb1] O. Kobayashi, Scalar curvature of a metric of unit volume, Math. Annalen, vol. 279, (1987), 253-265.

[Kb2] ——, A differential equation arising from the scalar curvature, Jour. Math. Soc. Japan, vol 34, (1982), 665-675.

[Kr] D. Kramers, H. Stephani, M. McCallum, E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge U. Press, Cambridge, (1980).

[Ks] N. Koiso, A decomposition of the space of Riemannian metrics on a manifold, Osaka Jour. Math., vol 16, (1979), 423-429.

[La] J. Lafontaine, Sur la géométrie d'une généralisation de l'équation différéntielle d'Obata, Jour. Math. Pures Appl., vol 62, (1983), 63-72.

[Li] A. Lichnerowicz, Théories Relativistes de la Gravitation et de L'Électromagnétisme, Masson, Paris, (1955).

[LP] J. Lee and T. Parker, The Yamabe problem, Bull. Amer. Math. Soc., 17:1, (1987), 37-91.

[M] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, Cambridge U. Press, Cambridge, U.K. (1995).

[Mo] J. Moser, On the volume element of a manifold, Trans. A.M.S., vol. 120, (1965), 286-294.

[PS] R. Palais and S. Smale, A generalized Morse theory, Bull. A.M.S., vol. 70, (1964), 165-171.

[Ro] D. Robinson, A simple proof of the generalization of Israel's theorem, Gen. Rel. and Grav., vol 8:8, (1977), 695-698.

[Sc1] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom., 20, (1984), 479-495.

[Sc2] ——, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Lect. Notes Math. vol. 1365, Springer Verlag, Berlin, (1987), 120-154.

[SY] R. Schoen and S.T. Yau, On the proof of the positive mass conjecture in general relativity, Comm. Math. Phys., vol. 65, (1979), 45-76.

[Sz] P. Szekeres, Multipole particles in equilibrium in general relativity, Physical Review, vol. 176:5, (1968), 1446-1450.

[Th1] W. Thurston, The Geometry and Topology of 3-Manifolds, (Preprint), Princeton, 1979.

[Th2] ——, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, Bulletin A.M.S., Vol. 6, (1982), 357-381.

[Tr] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa, 22, (1968), 265-274.

[Y] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J., 12, (1960), 21-37.

[Yu] S. T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math., vol. 28, (1975), 201-228.

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